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Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics

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Sophia Demoulini* David M.A. Stuart† Athanasios E. Tzavaras‡

Abstract

For the equations of elastodynamics with polyconvex stored energy, and some related simpler systems, we define a notion of dissipative measure-valued solution and show that such a solution agrees with a classical solution with the same initial data when such a classical solution exists. As an application of the method we give a short proof of strong convergence in the continuum limit of a lattice approximation of one dimensional elastodynamics in the presence of a classical solution. Also, for a system of conservation laws endowed with a positive and convex entropy, we show that dissipative measure-valued solutions attain their initial data in a strong sense after time averaging.

1 Introduction

In this article we consider the system of equations of elastodynamics, with a stored energy function which satisfies the condition of polyconvexity introduced in [4]. This system can be embedded into a symmetrizable hyperbolic system of conservation laws which admits a convex entropy([9, 8, 18]). Using this embedding, the existence of globally defined measure-valued solutions (that satisfy additional geometric properties involving the null Lagrangians) was proved in [9], using a method of variational approximation. The concept of measure-valued solution was introduced into the theory of conservation laws in [10], and then into the theory of the incompressible Euler equations in [11], after the development of Young measures and weak convergence methods for partial differential equations ([19, 12]). For several equations of mathematical physics it is currently the only notion of solution which is sufficiently broad to allow for a global existence theory. However there are no corresponding uniqueness theorems, and the framework of measure-valued solutions is clearly inadequate to distinguish those solutions which are physically relevant, and has to be supplemented with further structural conditions on the solutions. Clearly a minimal requirement for any new concept of solution is that it should agree with the classical solution when such exists, and more generally it is worthwhile to determine properties of classical solutions which carry over to the new class of solutions.

We consider the *dissipative measure-valued solutions* (see definitions 2.1, 2.4, 3.1 and 4.1), which form a sub-class of the measure-valued solutions which satisfy an averaged and integrated form of

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the entropy inequality (which allows for concentration effects in the L^p, $p < \infty$ setting). We prove that, when a classical solution is present, the dissipative measure-valued and the classical solution coincide. The method of proof is based on the idea of relative entropy and the format of weak-strong uniqueness that was introduced in the context of conservation laws in [7, 8]. The *measure-valued-strong uniqueness* which we prove here handles both oscillations and concentrations, and it is a further consequence of the method of proof that when a classical solution exists a dissipative measure-valued solution does not admit concentrations in the entropy. To carry out this generalization one needs to account for concentrations in the approximating sequence as in [11] and [1]. For present purposes however we do not need the general representation of concentrations obtained in these articles, because we only consider concentration effects for a single function - the entropy which appears in the definition of dissipative solution. In appendix A we provide a completely elementary derivation of the *Young measure with concentration* representation of the weak limit of this function, see (A.6).

The second issue we study in section 4 is the role of (the measure-valued form of) the entropy inequality and the sense in which *entropic measure-valued solutions* assume the initial data. Several authors have studied the problem of the initial trace of solutions for conservation laws, starting with [10] and then [6], [21] (using genuine nonlinearity) and [17] (exploiting the entropy inequality). We show that when the Young measure associated to the family of initial data is a Dirac mass a time average of a dissipative measure-valued solution converges strongly to the initial data (see theorem 4.3). This result, which extends the obervations of DiPerna in [10, section 6(e)] to an L^p context where there is the possibility of the development of concentrations which has to be eliminated, represents another noteworthy consequence of the convexity of the entropy.

The relative entropy method used here to prove measure-valued-strong uniqueness provides a clean and quick proof of strong convergence of approximation schemes to conservation laws in the time regime in which the conservation laws admit classical solutions. To explain this recall that a conservative view of measure-valued solutions is that they provide an efficient way of encoding some properties of weakly convergent approximating sequences to a system of equations. Once an approximation scheme is established which is stable, in the precise sense that it generates a dissipative measure-valued solution, measure-valued-strong uniqueness automatically implies strong convergence, *without energy concentration*, of the approximating sequence to the classical solution. We illustrate this aspect by considering a lattice approximation of the equations of elasticity (in one space dimension) by a system of point masses connected by nonlinear springs, and prove strong convergence of the spring-mass system to the equations of one-dimensional elastodynamics in the continuum limit (as long as the latter admits a classical solution).

After the completion of this work we became aware of a recent article by Brenier-DeLellis-Szèkelyhidi [5] in which weak-strong uniqueness is proved for measure-valued solutions of the Euler equations. Although the focus of our article is a different system of equations, with specific intrinsic features - notably the lack of uniform convexity and the embedding into the enlarged system (3.2)-(3.3) via the null Lagrangians - there is overlap both in terms of general ideology and more specifically of the material in section 2.1 on conservation laws with L^∞ bounds. Nevertheless, we retain this material for explanatory purposes.

The article is organized as follows: in section 2 we introduce the problem and then in section 2.1 we perform the basic relative entropy computation at the level of a system of conservation laws with L^∞ bounds for an approximating sequence, and deduce measure-valued-strong uniqueness. (theorem 2.2). Then in section 2.2 we generalize to handle the situation that the approximating sequence is only bounded in L^2: we study the quasi-linear wave equation with convex stored energy satisfying quadratic growth conditions above and below, and show how to handle the possibility of concentrations using the material in appendix A. In section 3 we recall the global existence of measure-valued solutions for polyconvex elastodynamics from [9] and show that they are dissipative (where the relevant entropy is the energy, re-interpreted as the convex entropy for the enlarged system (3.2)-(3.3)). We then show that the relative entropy computation can be performed for this system and prove measure-valued-strong uniqueness (theorem 3.3). In section 4 we discuss general systems of conservation laws with L^p bounds, first extending measure-valued-strong uniqueness to the L^p case in theorem 4.2 and then proving theorem 4.3 on the strong attainment of the initial data. Finally section 5 is on the lattice-continuum limit for one dimensional elastodynamics.

As a final comment, the embedding of polyconvex elastodynamics into (3.6)-(3.7) notwithstanding, theorem 3.3 is not a consequence of theorem 4.2 on general systems of conservation laws: both the statement of the hypotheses for and the proof of theorem 3.3 make use of specific structural features of polyconvexity and the proof requires the weak continuity of the null Lagrangians.

2 Relative entropy for measure-valued solutions

Consider the system of conservation laws,

$$\partial_t v + \operatorname{div}_x f(v) = 0\,, \tag{2.1}$$

where $v = (v_1, \dots v_n)$ are functions of $x = (x_1, \dots x_d) \in \mathbb{R}^d$ and $t \geq 0$. Attempts to prove an existence theorem for 2.1 typically involve the study of a sequence of functions v^ϵ which are solutions of an approximating problem

$$\partial_t v^\varepsilon + \operatorname{div}_x f(v^\varepsilon) = \mathcal{P}_\varepsilon \tag{2.2}$$

where $\mathcal{P}_\varepsilon \to 0$ in distributions. Uniform bounds for the sequence are typically a consequence of an entropy inequality for the appoximating problem:

$$\partial_t \eta(v^\varepsilon) + \operatorname{div}_x q(v^\varepsilon) \leq \mathcal{Q}_\varepsilon \tag{2.3}$$

with again $\mathcal{Q}_\varepsilon \to 0$ in distributions. Typically (2.3) provides the available uniform bounds, $\sup_{\epsilon,t} \int \eta(v^\epsilon(x,t))\,dx < \infty$, for the sequence of approximate solutions. In the limit such an approximation procedure typically yields a measure-valued solution verifying a measure-valued version of the entropy inequality. One technical difficulty arising here however is that classical Young measures represent weak limits of functions of growth strictly less than that of η but are insufficient to represent the weak limit of η itself. The class of Young measures has to be adapted to reflect the representation of the weak limits of the entropy function in the presence of concentrations. We present a self-contained development of a technical tool designed to address this difficulty in appendix A, see (A.6). The concentration measure developed there (see (A.6)) will be incorporated in the definition of the class of dissipative measure-valued solutions which are studied in this article.

In this section we explain in the context of two model problems how to prove that, in the presence of a classical solution, a dissipative measure-valued solution with the same initial data necessarily agrees with that classical solution (*measure-valued-strong uniqueness*). The presentation is split into two: in section 2.1, in the presence of uniform L^∞ bounds, classical Young measures are used for the definition of measure-valued solution and the basic relative entropy computation ([8, Section

5.2]) is shown to extend to the measure-valued situation, yielding the proof of theorem 2.2. In section 2.2, we take up a model problem for the equations of elastodynamics: the quasi-linear wave equation with convex quadratic stored energy, where the appropriate stability framework involves uniform L^2 bounds. There, the tool of Young measure with energy concentration developed in appendix A is used to define the appropriate notion of *dissipative* measure-valued solution, and this is then used to prove theorem 2.5 on measure-valued-strong uniqueness in the presence of energy concentration.

2.1 Conservation laws with L^∞ bounds

Consider the system (2.1) written in coordinate form,

$$\frac{\partial v_j}{\partial t} + \frac{\partial f_{j\alpha}}{\partial x_\alpha} = 0, , \tag{2.4}$$

where latin indices $i, j, k \ldots$ are used for the target and greek indices $\alpha, \beta \ldots$ for the domain. The summation convention will be used throughout. To avoid inessential issues, we will work in the spatially periodic case and spatial integrals will be over the fundamental domain of periodicity $Q = (\mathbb{R}/2\pi\mathbb{Z})^d$. We write $Q_T = Q \times [0, T)$ for $T \in [0, +\infty)$ and $\overline{Q}_T = Q \times [0, T]$.

We assume that (2.4) is endowed with an entropy - entropy flux pair $\eta - q$, that is, it is equipped with an additional conservation law

$$\frac{\partial \eta}{\partial t} + \frac{\partial q_\alpha}{\partial x_\alpha} = 0\,, \tag{2.5}$$

and that the entropy function η is convex. Then, $\eta - q$ satisfy the consistency equations

$$\frac{\partial \eta}{\partial v_j} \frac{\partial f_{j\alpha}}{\partial v_i} = \frac{\partial q_\alpha}{\partial v_i}\,, \tag{2.6}$$

or equivalently

$$\frac{\partial^2 \eta}{\partial v_k \partial v_j} \frac{\partial f_{j\alpha}}{\partial v_i} = \frac{\partial^2 \eta}{\partial v_i \partial v_j} \frac{\partial f_{j\alpha}}{\partial v_k}. \tag{2.7}$$

All functions f, η, q are assumed C^2 and we assume positivity of the Hessian matrix $\nabla^2 \eta$ (which implies strict convexity of η).

Definition 2.1 Let $\boldsymbol{\nu} = \{\boldsymbol{\nu}_{x,t}\}_{\{(x,t)\in\overline{Q}_T\}}$ be a parametrized family of probability measures that are all supported within a compact subset $D \subset \mathbb{R}^n$, and with the property that for all continuous $f : \mathbb{R}^n \to \mathbb{R}$

$$\langle \boldsymbol{\nu},\, f\, \rangle = \langle \boldsymbol{\nu}_{x,t},\, f\, \rangle = \int f(\lambda)\, d\boldsymbol{\nu}(\lambda)$$

is a measurable function of (x, t).

(i) The pair $(v, \boldsymbol{\nu})$ is a *measure-valued solution* of (2.4) with initial values $v_0(x)$, if it verifies $v = \int \lambda\, d\boldsymbol{\nu}(\lambda) \in L^\infty(dxdt)$ and

$$\iint \left[\frac{\partial \psi_i}{\partial t} v_i + \frac{\partial \psi_i}{\partial x_\alpha} \langle \boldsymbol{\nu}, f_{i\alpha} \rangle \right] dxdt + \int \psi_i(x, 0) v_{0,i}(x) dx = 0, \tag{2.8}$$

for any test functions $\psi = \psi(x, t) \in C^1_c(Q_T)$.

(ii) It will be called an *entropic* measure-valued solution of (2.4) if, in addition, for *non-negative* test functions, $\psi \in C_c^1\big(Q_T\big)$ with $\psi \geq 0$, there holds:

$$\iint \left[\frac{\partial \psi}{\partial t} \langle \boldsymbol{\nu}, \eta \rangle + \frac{\partial \psi}{\partial x_\alpha} \langle \boldsymbol{\nu}, q_\alpha \rangle \right] dxdt + \int \psi(x,0)\eta(v_0(x))dx \geq 0. \tag{2.9}$$

(iii) It will be called a *dissipative* measure-valued solution if this inequality holds only for non-negative test functions $\psi(x,t) = \theta(t)$ depending solely on time, i.e. if

$$\iint \frac{d\theta}{dt} \langle \boldsymbol{\nu}, \eta \rangle dxdt + \int \theta(0)\eta(v_0(x))dx \geq 0. \tag{2.10}$$

for all $\theta \in C_c^1\big([0,T)\big)$ satisfying $\theta \geq 0$.

We assume that there is a classical solution of (2.4) on $\overline{Q}_T$, to be precise a function $\overline{v} \in W^{1,\infty}(\overline{Q}_T)$ (i.e. a bounded function which is differentiable a.e. *with bounded derivative*) which verifies the strong (or classical) versions of (2.8) and (2.10):

$$\iint \left[\frac{\partial \psi_i}{\partial t} \overline{v}_i + \frac{\partial \psi_i}{\partial x_\alpha} f_{i\alpha}(\overline{v})\rangle \right] dxdt + \int \psi_i(x,0)\overline{v}_{0,i}(x)dx = 0\,, \tag{2.11}$$

and

$$\iint \frac{d\theta}{dt} \eta(\overline{v})\, dxdt + \int \theta(0)\eta(\overline{v}_0(x))dx = 0\,, \tag{2.12}$$

for all test functions ψ, θ as above. (Note that (2.12) is now an equality). In this circumstance we have the following:

Theorem 2.2 *Let $\overline{v} \in W^{1,\infty}(\overline{Q}_T)$ verify* (2.11) *and* (2.12) *and let $(v, \boldsymbol{\nu})$ be a dissipative measure-valued solution verifying* (2.8) *and* (2.10)*. Assume there exists a compact set $D \subset \mathbb{R}^n$ in which $\overline{v}$ takes its values, and assume also that v takes its values in D, and that $\boldsymbol{\nu}$ is supported in D. Then there exists $c_1 > 0, c_2 > 0$ such that for $t \in [0,T]$:*

$$\iint |\lambda - \overline{v}|^2 \, d\boldsymbol{\nu}(\lambda)dx \leq c_1 \left(\int |v_0 - \overline{v}_0|^2 \, dx \right) e^{c_2 t}\,, \tag{2.13}$$

and in particular if the initial data agree, $v_0 = \overline{v}_0$ then $\boldsymbol{\nu} = \delta_{\overline{v}}$ and $v = \overline{v}$ almost everywhere.

Proof Introduce the *relative entropy*

$$\eta_{rel}(\lambda, \overline{v}) := \eta(\lambda) - \eta(\overline{v}) - \frac{\partial \eta}{\partial v_j}(\overline{v})(\lambda_j - \overline{v}_j)\,, \tag{2.14}$$

the averaged quantities

$$h(\boldsymbol{\nu}, v, \overline{v}) := \langle \boldsymbol{\nu}, \eta \rangle - \eta(\overline{v}) - \frac{\partial \eta}{\partial v_j}(\overline{v})(v_j - \overline{v}_j)\,, \tag{2.15}$$

$$Z_{k\alpha}(\boldsymbol{\nu}, v, \overline{v}) := \langle \boldsymbol{\nu}, f_{k\alpha} \rangle - f_{k\alpha}(\overline{v}) - \frac{\partial f_{k\alpha}}{\partial v_j}(\overline{v})(v_j - \overline{v}_j)\,, \tag{2.16}$$

and note that, since $\boldsymbol{\nu}$ is a probability measure at each x, t, it is possible to write

$$h(\boldsymbol{\nu}, v, \overline{v}) = \int \Big(\eta(\lambda) - \eta(\overline{v}) - \frac{\partial \eta}{\partial v_j}(\overline{v})(\lambda_j - \overline{v}_j) \Big)\, d\boldsymbol{\nu}(\lambda) = \int \eta_{rel}(\lambda, \overline{v})\, d\boldsymbol{\nu}(\lambda)\,. \tag{2.17}$$

Next, using (2.4) and (2.7) we calculate that:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial v_j}(\overline{v}) \right) = \frac{\partial \overline{v}_k}{\partial t} \frac{\partial^2 \eta}{\partial v_k \partial v_j}(\overline{v}) &= -\frac{\partial}{\partial x_\alpha} f_{k\alpha}(\overline{v}) \frac{\partial^2 \eta}{\partial v_k \partial v_j}(\overline{v}) \\
&= -\frac{\partial \overline{v}_l}{\partial x_\alpha} \frac{\partial f_{k\alpha}}{\partial v_l}(\overline{v}) \frac{\partial^2 \eta}{\partial v_k \partial v_j}(\overline{v}) \\
&= -\frac{\partial \overline{v}_l}{\partial x_\alpha} \frac{\partial f_{k\alpha}}{\partial v_j}(\overline{v}) \frac{\partial^2 \eta}{\partial v_k \partial v_l}(\overline{v})\,, \qquad \text{by (2.7)}\,.
\end{aligned}$$

Since this is a bounded function (on account of the hypothesis that $\overline{v}$ is Lipschitz), and referring to the definition of Z in (2.16) above, we deduce that

$$\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial v_j}(\overline{v}) \right)(v_j - \overline{v}_j) + \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \eta}{\partial v_k}(\overline{v}) \right)\Big(\langle \boldsymbol{\nu}, f_{k\alpha} \rangle - f_{k\alpha}(\overline{v}) \Big) = \frac{\partial \overline{v}_l}{\partial x_\alpha}\left(\frac{\partial^2 \eta}{\partial v_k \partial v_l}(\overline{v}) \right) Z_{k\alpha}. \tag{2.18}$$

Note that, upon using an approximation argument, ψ and θ in (2.8), (2.9), (2.10), (2.11) and (2.12) can be taken to be Lipschitz functions that vanish for sufficiently large times. Now choose $\psi(x,t) = \theta(t) \frac{\partial \eta}{\partial v_j}(\overline{v}(x,t))$ in (2.8) and (2.11), subtract them, and then apply (2.18) to get:

$$\iint \left[\frac{d\theta}{dt} \frac{\partial \eta}{\partial v_j}(\overline{v})(v_j - \overline{v}_j) + \theta \frac{\partial \overline{v}_l}{\partial x_\alpha}\left(\frac{\partial^2 \eta}{\partial v_k \partial v_l}(\overline{v}) \right) Z_{k\alpha} \right] dx dt + \int \theta \frac{\partial \eta}{\partial v_j}(\overline{v}) \bigg|_{t=0} \big[v_{0,j}(x) - \overline{v}_{0,j}(x) \big]\, dx = 0.$$

Next, subtract this equation from (2.10), and also subtract (2.12), leading to:

$$\begin{aligned}
\iint \dot{\theta}\, h\, dx d\tau \geq \iint \theta \frac{\partial \overline{v}_l}{\partial x_\alpha}\left(\frac{\partial^2 \eta}{\partial v_k \partial v_l}(\overline{v}) \right) Z_{k\alpha}\, dx d\tau \\
- \int \theta(0) \Big[\eta(v_0) - \eta(\overline{v}_0) - \frac{\partial \eta}{\partial v_i}(\overline{v}_0)(v_0 - \overline{v}_0)_i \Big] dx,
\end{aligned} \tag{2.19}$$

for non-negative Lipschitz test functions $\theta = \theta(\tau)$. Now let $\theta(\tau)$ be the non-negative piecewise linear function given by

$$\theta(\tau) \equiv \begin{cases} 1 \ \text{ when } \ 0 \leq \tau < t\,, \\ 0 \ \text{ when } \ \tau \geq t + \epsilon\,, \\ \frac{t-\tau}{\epsilon} + 1 \quad \text{when} \quad t \leq \tau < t + \epsilon\,. \end{cases} \tag{2.20}$$

With this choice of θ (2.19) reads

$$-\frac{1}{\epsilon} \int_t^{t+\epsilon} \int h dx d\tau \geq \iint \theta(\tau) \frac{\partial \overline{v}_i}{\partial x_\alpha} \frac{\partial^2 \eta(\overline{v})}{\partial v_k \partial v_i} Z_{k\alpha} dx d\tau - \int \Big[\eta(v_0) - \eta(\overline{v}_0) - \frac{\partial \eta}{\partial v_i}(\overline{v}_0)(v_0 - \overline{v}_0)_i \Big] dx \tag{2.21}$$

which implies, in the limit $\epsilon \to 0$, that

$$\int h\, dx \leq c \int_0^t \int \max_{k,\alpha} |Z_{k\alpha}|\, dx\, d\tau + \int \Big[\eta(v_0) - \eta(\overline{v}_0) - \frac{\partial \eta}{\partial v_i}(\overline{v}_0)(v_0 - \overline{v}_0)_i \Big] dx \tag{2.22}$$

for $t \in (0,T)$.

Under the working assumption that η has strictly positive second derivative, there exists $c_0 = c_0(D) > 0$ such that

$$h(\boldsymbol{\nu}, v, \overline{v}) \geq c_0 \int |\lambda - \overline{v}|^2 d\boldsymbol{\nu}(\lambda). \tag{2.23}$$

Notice also that, for some $C = C(D)$,

$$\begin{aligned} |Z_{k\alpha}(\boldsymbol{\nu}, v, \overline{v})| &= |\langle \boldsymbol{\nu}, f_{k\alpha}(\lambda) - f_{k\alpha}(\overline{v}) - \frac{\partial f_{k\alpha}}{\partial v_j}(\overline{v})(\lambda_j - \overline{v}_j)\rangle| \\ &\leq C \int |\lambda - \overline{v}|^2 d\boldsymbol{\nu}(\lambda)\,. \end{aligned} \tag{2.24}$$

Hence,

$$\begin{aligned} c_0 \int |\lambda - \overline{v}|^2 \, d\boldsymbol{\nu}(\lambda)\, dx &\leq \int h \, dx \\ &\leq c \int_0^t \int \int |\lambda - \overline{v}|^2 \, d\boldsymbol{\nu}(\lambda)\, dx\, d\tau + c' \int |v_0 - \overline{v}_0|^2 \, dx\,, \end{aligned} \tag{2.25}$$

where $c = c(D, |\overline{v}|_{W^{1,\infty}})$ and $c' = c'(D)$. Therefore Gronwall's inequality implies the bound (2.13) and the fact that if $\overline{v}_0 = v_0$ then $\int |\lambda - \overline{v}|^2 \, d\boldsymbol{\nu}(\lambda)\, dx$ is zero at later times, i.e. the measure-valued solution agrees with the classical solution $\overline{v}$ almost everywhere. □

The above calculation is a measure-valued version of the calculation in [8, Section 5.2]. In an analogous fashion, it can be carried through for test functions with more general x-dependence to give a measure valued version of equation (5.2.6) in that reference, but we do not pursue that here.

2.2 Quasilinear wave equation with convex energy and L^2 bounds

In this section we consider the quasi-linear wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \nabla \cdot S(\nabla y), \tag{2.26}$$

where $y : Q \times \mathbb{R}^+ \to \mathbb{R}^3$ and S is the gradient of a strictly convex function $G : \mathrm{Mat}^{3\times 3} \to [0, \infty)$, about which we make the following hypotheses:

(a1) $G \in C^3$ and $m|Z|^2 \leq D^2G(\hat{F})[Z,Z] \leq M|Z|^2$;

(a2) $G(F) = g_0(F) + \frac{1}{2}|F|^2$ where $\lim_{|F|\to\infty} \frac{g_0(F)}{1+|F|^2} = 0$.

(a3) $\lim_{|F|\to\infty} \frac{|\nabla_F G(F)|}{1+|F|^2} = 0$

(a4) $|D^3G(F)| \leq M$, for some $M > 0$.

(We use the summation convention for repeated indices, the norm $|F|^2 = F_{i\alpha}F_{i\alpha}$ and explicitly the second derivative is given by $D^2G(\hat{F})[Z,\tilde{Z}] = \frac{\partial^2 G(\hat{F})}{\partial F_{i\alpha}\partial F_{j\beta}} Z_{i\alpha}\tilde{Z}_{j\beta}$.) If y is interpreted as a displacement vector this equation could be regarded as a model for elastodynamics, but the assumption of convexity is known to be physically unrealistic. We consider a more realistic model in section 3.

A classical solution of (2.26) means a C^1 function whose first derivatives are Lipschitz and verify (2.26) almost everywhere. Alternatively, introducing the notation $v_i = \partial_t y_i$ and $F_{i\alpha} = \frac{\partial y_i}{\partial x_\alpha}$, a classical solution to (2.26) in first order form consists of a pair (v, F) of Lipschitz functions which solve

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial G}{\partial F_{i\alpha}}\right) \tag{2.27}$$

$$\frac{\partial F_{i\alpha}}{\partial t} = \frac{\partial v_i}{\partial x_\alpha}. \tag{2.28}$$

Such a solution will automatically satisfy the conservation law

$$\partial_t \eta + \partial_\alpha q_\alpha = 0 \tag{2.29}$$

where $\eta(v, F) = \frac{1}{2}|v|^2 + G(F)$ and $q_\alpha(v, F) = v_i \frac{\partial G}{\partial F_{i\alpha}}(F)$, and take on the initial data $v^0(x) = v(0, x)$ and $F^0(x) = F(0, x)$ in the uniform norm.

Definition 2.3 A *measure-valued* solution to (2.26) with initial data $(v^0, F^0) \in L^2 \oplus L^2$ consists of a pair $(v, F) \in L^\infty(L^2) \oplus L^\infty(L^2)$ and a Young measure $\boldsymbol{\nu} = (\boldsymbol{\nu}_{x,t})_{x,t\in\overline{Q}_T}$ generated by a sequence satisfying (2.32) such that for $i, \alpha = 1, \dots 3$

$$\int \psi(0,x) v_i^0(x)\, dx + \iint v_i \partial_t \psi \; dxdt = \iint \left\langle \boldsymbol{\nu}, \frac{\partial G}{\partial F_{i\alpha}} \right\rangle \partial_\alpha \psi \; dxdt \tag{2.30}$$

$$\int \psi(0,x) F_{i\alpha}^0(x)\, dx + \iint F_{i\alpha} \partial_t \psi dxdt = \iint v_i \partial_\alpha \psi \, dxdt \tag{2.31}$$

for all test functions $\psi = \psi(t, x) \in C^1_c(Q_T)$.

In order to define a sense in which a measure-valued solution satisfies the entropy condition (2.29) as an inequality, it is necessary to introduce some method of describing concentration effects in sequences of approximate solutions. Any natural construction of a measure-valued solution to (2.27)-(2.28), e.g. by the viscosity method or by time-discretization, produces a family of functions (v^ϵ, F^ϵ) of uniformly bounded energy:

$$\sup_\epsilon \sup_{t\geq 0} \int \eta(v^\epsilon, F^\epsilon)\, dx \; < \; +\infty \tag{2.32}$$

which are therefore bounded in $L^\infty(L^2) \oplus L^\infty(L^2)$. Weak limits of such approximate solutions limit must be represented somehow. For functions of (v, F) of growth at infinity strictly less than quadratic the ordinary Young measure as developed in [2] is sufficient, providing a weakly measurable family of probability measures which represent weak limits of functions of (v^ϵ, F^ϵ) which are weakly precompact in L^1. On the other hand, in order to discuss the weak limit of quadratic quantities such as $\eta(v^\epsilon, F^\epsilon)$ it is necessary to describe any limiting concentration formations in the sequences. In appendix A we introduce a *non-negative* Radon measure $\boldsymbol{\gamma}$ to measure concentration effects in the energy

$$\boldsymbol{\gamma}(\psi) = \iint \psi(x,t) \boldsymbol{\gamma}(dxdt) = \frac{1}{2} \lim_{\epsilon\to 0} \iint \psi \left(|v^\epsilon|^2 - \langle \boldsymbol{\nu}_{x,t}, |\lambda|^2 \rangle + |F^\epsilon|^2 - \langle \boldsymbol{\nu}_{x,t}, |M|^2 \rangle \right) dxdt\,, \tag{2.33}$$

for all bounded continuous ψ vanishing for large times, see (A.4). (Here $\boldsymbol{\nu}_{x,t}$ is a probability measure on $\mathbb{R}^3 \times \mathrm{Mat}^{3\times 3}$, and we write (λ, M) for the coordinates on $\mathbb{R}^3 \times \mathrm{Mat}^{3\times 3}$ used in the integration with respect to the measure $\boldsymbol{\nu}$.) For the class of nonlinear energies G under consideration we will then have by the Young measure representation (subsequentially):

$$\iint \psi\, \eta(v^\epsilon, F^\epsilon)\, dxdt \;\rightarrow\; \iint \psi\big(\langle \boldsymbol{\nu}_{x,t}, \eta\rangle\, dxdt + \boldsymbol{\gamma}(dxdt)\big)\,, \tag{2.34}$$

for all such ψ.

The approximate solutions (v^ϵ, F^ϵ) are generated by families of initial data

$$(v^{\epsilon,0}(x), F^{\epsilon,0}(x)) = (v^\epsilon(0,x), F^\epsilon(0,x))\,, \tag{2.35}$$

converging weakly in L^2 to $(v^0(x), F^0(x))$. According to the results of section A.1, the initial data generate a Young measure $\boldsymbol{\mu}_x$ and an energy concentration measure $\boldsymbol{\zeta}(dx)$ with the property that (along subsequences)

$$\int \phi(x) g(v^{\epsilon,0}, F^{\epsilon,0})\, dx \rightarrow \int \phi(x)\langle \boldsymbol{\mu}_x, g(\lambda, M)\rangle\, dx \tag{2.36}$$

for all continuous ϕ and subquadratic g, and

$$\int \phi(x)\eta(v^{\epsilon,0}, F^{\epsilon,0})\, dx \rightarrow \int \phi(x)\langle \boldsymbol{\mu}_x, \eta(\lambda, M)\rangle\, dx + \int \phi(x)\boldsymbol{\zeta}(dx) \tag{2.37}$$

for all continuous ϕ. In this situation we shall refer to *Young measure initial data* $(v^0, F^0, \boldsymbol{\mu}, \boldsymbol{\zeta})$ for brevity. The important special case that the initial data converge strongly corresponds to $\boldsymbol{\zeta} \equiv 0$ and to the Young measure $\boldsymbol{\mu}_x$ being a Dirac measure. In the definition of measure valued solutions we think of fixed initial data, or sequences of data that converge strongly, i.e. $\boldsymbol{\mu}_x$ being a Dirac measure. The definition can be easily adjusted to accomodate more general situations.

Assume now that (v^ϵ, F^ϵ) is a sequence bounded in $L^\infty(L^2) \oplus L^\infty(L^2)$, verifying (2.35)-(2.37), which generates the measure-valued solution verifying (2.30)-(2.31), and the entropy inequality

$$\int \psi(0,x)\eta(v^{\epsilon,0}, F^{\epsilon,0})dx + \iint \partial_t\psi\eta(v^\epsilon, F^\epsilon) + \partial_\alpha\psi\, q_\alpha(v^\epsilon, F^\epsilon)dxdt \geq 0\,,$$

for $\psi \in C^1(Q_T)$. Taking the limit $\epsilon \rightarrow 0$ and using (2.34), (2.37) (with $\boldsymbol{\mu}_x$ a Dirac measure, $\boldsymbol{\zeta} \equiv 0$) motivates the following definition of dissipative measure-valued solution:

Definition 2.4 Given initial data $(v^0, F^0) \in L^2 \oplus L^2$ a *dissipative measure-valued solution with concentration* to (2.27)-(2.28) and (2.29) consists of a pair $(v, F) \in L^\infty(L^2) \oplus L^\infty(L^2)$, a Young measure $\boldsymbol{\nu} = (\boldsymbol{\nu}_{x,t})_{x,t\in\overline{Q}_T}$ and a non-negative Radon measure $\boldsymbol{\gamma} \in \mathcal{M}^+(Q_T)$ such that $(v, F, \boldsymbol{\nu})$ is a measure-valued solution verifying (2.30)-(2.31), and in addition:

$$\iint \frac{d\theta}{dt}\big(\langle \boldsymbol{\nu}_{x,t}, \eta\rangle\, dxdt\, + \boldsymbol{\gamma}(dxdt)\big) + \int \theta(0)\eta(v^0, F^0)\, dx \geq 0\,, \tag{2.38}$$

for all non-negative functions $\theta(t) \in C^1_c([0,T))$.

Theorem 2.5 *Consider a dissipative measure-valued solution with concentration to* (2.27)-(2.28) *as just defined, associated to initial data* (v^0, F^0).

(i) If $(\hat{v},\hat{F})\in W^{1,\infty}(\overline{Q}_T)$ is a Lipschitz classical solution with initial data $(\hat{v}^0,\hat{F}^0)$, there exist $c_1,c_2>0$ such that for $0\le t\le T$:

$$\int \langle \boldsymbol{\nu}, |\lambda-\hat{v}|^2+|M-\hat{F}|^2\rangle\,dx \le c_1\Big(\int |v^0-\hat{v}^0|^2+|F^0-\hat{F}^0|^2\,dx\Big)\,e^{c_2 t}\,. \tag{2.39}$$

(ii) If in addition $v^0=\hat{v}^0$ and $F^0=\hat{F}^0$ almost everywhere, then $(v,F)=(\hat{v},\hat{F})$, and $\boldsymbol{\nu}_{x,t}=\delta_{\hat{v}(x,t),\hat{F}(x,t)}$ almost everywhere and the concentration measure $\boldsymbol{\gamma}$ is null in Q_T.

Proof Let $(v,F,\boldsymbol{\nu},\boldsymbol{\gamma})$ be a dissipative measure-valued solution satisfying (2.30), (2.31) and (2.38). We note that using an approximation argument (2.30)-(2.31) can be extended to hold for Lipschitz test functions ψ that vanish for large times: here we use the assumpion that $\boldsymbol{\nu}$ is generated by a sequence verifying (2.32) which ensures that all quantities in (2.30)-(2.31) lie in L^1 under the hypotheses (a1)-(a4) and so the bounded convergence theorem applies. By contrast, (2.38) cannot be extended to this class in the absence of further infomation about the concentration measure $\boldsymbol{\gamma}$.

Assume that $(\hat{v},\hat{F})$ is a classical solution as defined above. It will satisfy (2.38) as an equality:

$$\iint \frac{d\theta}{dt}\,\langle \boldsymbol{\nu}_{x,t},\hat{\eta}\rangle\,dxdt\,+\int \theta(0)\hat{\eta}_0(x)dx=0\,, \tag{2.40}$$

where $\hat{\eta}=\eta(\hat{v},\hat{F})$ is the energy evaluated along the solution. Now subtracting from (2.30)-(2.31) the corresponding equations for the classical solution $(\hat{v},\hat{F})$, and choosing the test functions in the resulting equations to be, respectively, $\theta(t)\hat{v}_i$, and $\theta(t)\frac{\partial G}{\partial F_{i\alpha}}(\hat{F})$, where θ is a C^1 function of time vanishing for sufficiently large times, we obtain the following identity:

$$\begin{aligned}
&\int \theta(0)\,\hat{v}_i(0,x)(v_i-\hat{v}_i)(0,x)\,dx+\int \theta(0)\,\frac{\partial G}{\partial F_{i\alpha}}(\hat{F}_{i\alpha}(0,x))\big(F_{i\alpha}(0,x)-\hat{F}_{i\alpha}(0,x)\big)\,dx\\
&\quad+\iint \Big[(v_i-\hat{v}_i)\hat{v}_i+(F_{i\alpha}-\hat{F}_{i\alpha})\frac{\partial G}{\partial F_{i\alpha}}(\hat{F})\Big]\,\partial_t\theta\,dxdt\\
&\qquad=\iint \theta\,(\partial_\alpha\hat{v}_i)\Big\langle \boldsymbol{\nu}_{x,t}\,,\frac{\partial G(M)}{\partial F_{i\alpha}}-\frac{\partial G(\hat{F})}{\partial F_{i\alpha}}-\frac{\partial^2 G(\hat{F})}{\partial F_{i\alpha}\partial F_{j\beta}}(M_{j\beta}-\hat{F}_{j\beta})\Big\rangle\,dxdt\;\equiv\mathcal{Q}
\end{aligned} \tag{2.41}$$

This calculation is very similar, but simpler, to one given in full in the next section, and so will not be written out.

Define the relative entropy as

$$\eta_{rel}(\lambda,M;\hat{v},\hat{F})\equiv\frac{1}{2}|\lambda-\hat{v}|^2+G(M)-G(\hat{F})-\frac{\partial G(\hat{F})}{\partial F_{i\alpha}}(M_{i\alpha}-\hat{F}_{i\alpha})\,, \tag{2.42}$$

and its $t=0$ version as

$$\eta_{rel,0}=\eta_{rel}(\lambda,M;\hat{v}^0,\hat{F}^0)\equiv\frac{1}{2}|\lambda-\hat{v}^0|^2+G(M)-G(\hat{F}^0)-\frac{\partial G(\hat{F})}{\partial F_{i\alpha}}(M_{i\alpha}-\hat{F}^0_{i\alpha})\,. \tag{2.43}$$

Hypotheses (a1) and (a2) guarantee that η_{rel} (resp. $\eta_{rel,0}$) are bounded above and below by multiples of $|\lambda-\hat{v}|^2+|M-\hat{F}|^2$ (resp. $|\lambda-\hat{v}^0|^2+|M-\hat{F}^0|^2$). Combining (2.38), (2.40) and (2.41), we obtain

$$\iint \dot{\theta}\,\big(\langle \boldsymbol{\nu}_{x,\tau},\eta_{rel}(\lambda,M;\hat{v},\hat{F})\rangle\,dxd\tau\,+\boldsymbol{\gamma}(dxd\tau)\big)+\,\theta(0)\int \eta_{rel}(v^0,F^0;\hat{v}^0,\hat{F}^0)\,dx\ge-\mathcal{Q}\,, \tag{2.44}$$

where $\theta = \theta(\tau) \in C_c^1([0,T))$. We would like to choose θ as in (2.20), but this is not C^1. Therefore we choose a sequence of functions $\theta^n \in C_c^1([0,T))$ which are bounded (uniformly in n), non-increasing and have the property that $\dot\theta^n(\tau) \to \dot\theta(\tau)$ for $\tau \neq t, t+\epsilon$. Since $\dot\theta^n \le 0$ and $\boldsymbol{\gamma} \ge 0$, we can discard $\dot\theta^n \boldsymbol{\gamma}$ in the inequality (2.44). Referring to (2.41) and substituting in $\theta^n(\tau)$, we use assumption (a4). to deduce that there exists $C_1 = C_1(|\hat v|_{W^{1,\infty}})$ such that for all n

$$
|\mathcal{Q}| \le C_1 \int_0^{t+\epsilon} \int \langle \boldsymbol{\nu}_{x,t}\,, |M - \hat F|^2 \rangle \, dx d\tau \,. \tag{2.45}
$$

To take the limit $n \to \infty$, note that $\dot\theta^n$ are bounded and so are $\int \langle \boldsymbol{\nu}_{x,\tau}, \eta_{rel}(\lambda, M; \hat v, \hat F) \rangle \, dx$ (by the assumption on the generation of $\boldsymbol{\nu}$ by a sequence verifying (2.32)) so that by bounded convergence the time integrals converge. We obtain

$$
\frac{1}{\epsilon} \int_t^{t+\epsilon} \int \langle \boldsymbol{\nu}_{x,\tau}, \eta_{rel} \rangle \, dx \, d\tau \le \int \eta_{rel}(v^0, F^0; \hat v^0, \hat F^0) \, dx + C_1 \int_0^{t+\epsilon} \int \langle \boldsymbol{\nu}_{x,t}\,, |M - \hat F|^2 \rangle \, dx d\tau \,.
$$

Assumptions (a1) and (a2) imply that $\langle \boldsymbol{\nu}_{x,\tau}, \eta_{rel} \rangle \ge \frac{1}{C_2} \langle \boldsymbol{\nu}_{x,\tau}, |\lambda - \hat v|^2 + |M - \hat F|^2 \rangle$ for some $C_2 > 0$. Consider the function $\mathrm{Var}(\tau) = \int \langle \boldsymbol{\nu}_{x,\tau}, |\lambda - \hat v|^2 + |M - \hat F|^2 \rangle \, dx$, which is an averaged variance of the Young measure; it satisfies

$$
\frac{1}{\epsilon C_2} \int_t^{t+\epsilon} \mathrm{Var}(\tau) \, d\tau \le \int \eta_{rel}(v^0, F^0; \hat v^0, \hat F^0) \, dx + C_1 \int_0^{t+\epsilon} \mathrm{Var}(\tau) \, d\tau \,.
$$

Using Lebesgue's theorem, in the limit $\varepsilon \to 0$, $\mathrm{Var}(t)$ satisfies

$$
\mathrm{Var}(t) \le C_2 \int \eta_{rel}(v^0, F^0; \hat v^0, \hat F^0) \, dx \, dx \; + \; C_1 C_2 \int_0^t \mathrm{Var}(\tau) \, d\tau \,,
$$

for almost every $t \in (0,T)$. Therefore by Gronwall's inequality

$$
\mathrm{Var}(t) \le C_2 e^{C_1 C_2 t} \int \eta_{rel}(v^0, F^0; \hat v^0, \hat F^0) \, dx \,.
$$

In particular, if the initial data $(v^0, F^0) = (\hat v^0, \hat F^0)$ a.e. then the right hand side vanishes, the Young measure has zero variance for almost every x, t, and $\boldsymbol{\nu}_{x,t} = \delta_{\hat v(x,t), \hat F(x,t)}$. Going back to (2.38) we deduce that $\iint \dot\theta \boldsymbol{\gamma}(dxdt) \ge 0$ for all $\theta \in C_c^1([0,T))$ with $\theta \ge 0$ and so the concentration measure $\boldsymbol{\gamma} \ge 0$ is in fact identically zero. □

Remark 2.6 In writing down (2.30) in definition 2.3 the assumption (a3) is used in order to represent the weak limit of the stress. The situation should be contrasted to the Euler equations, where the flux is of the same order as the energy and the description of concentrations enters in the definition of measure-valued solutions, see Diperna-Majda [11].

3 Polyconvex elastodynamics

In this section we consider the system of elasticity

$$
\frac{\partial^2 y}{\partial t^2} = \nabla \cdot S(\nabla y), \tag{3.1}
$$

where $y : Q \times \mathbb{R}^+ \to \mathbb{R}^3$ stands for the motion, $F = \nabla y$, $v = \partial_t y$, and S stands for the Piola-Kirchoff stress tensor obtained as the gradient of a stored energy function, $S = \frac{\partial W}{\partial F}$. Here we assume that W is polyconvex, that is $W(F) = G(\Phi(F))$ where $G : \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \to [0,\infty)$ is a strictly convex function and $\Phi(F) = (F, \operatorname{cof} F, \det F) \in \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R}$ stands for the vector of null-Lagrangians: F, the cofactor matrix $\operatorname{cof} F$ and the determinant $\det F$.

We recall certain formal properties of the equations of polyconvex elasticity referring to [18, 8, 9] for details. Smooth solutions of (3.1) satisfy the system of conservation laws

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial G}{\partial \Xi^A}(\Phi(F))\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \tag{3.2}$$

$$\frac{\partial \Phi^A(F)}{\partial t} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right). \tag{3.3}$$

In checking this it is necessary to make use of the fact that the null-Lagrangians $\Phi(F)$ satisfy

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) = 0\,. \tag{3.4}$$

Given this, (3.3) follows from the chain rule and the formulae [9, (2.12-2.13)] for the derivatives of the null Lagrangians. In writing the above relations it is implicitly assumed that F is a gradient (which, if it holds initially, is a consequence of $\partial_t F = \nabla_x v$, and this equation is included as the first part of (3.3) since the components of F constitute the first nine components of $\Phi(F)$). Smooth solutions of (3.2)-(3.3) automatically satisfy the conservation of mechanical energy

$$\partial_t\Big(\frac{1}{2}|v|^2 + G(\Phi(F))\Big) - \partial_\alpha\Big(v_i\,\frac{\partial G}{\partial \Xi^A}(\Phi(F))\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\Big) = 0\,. \tag{3.5}$$

Using these observations the equations of polyconvex elasticity can be embedded into a symmetrizable hyperbolic system that determines the evolution of an enlarged vector $\Xi = (F, Z, w)$ taking values in $\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R}$ and treated as a new dependent variable:

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial G}{\partial \Xi^A}(\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \tag{3.6}$$

$$\frac{\partial \Xi^A}{\partial t} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right). \tag{3.7}$$

Smooth evolutions of this system preserve the constraints $\Xi^A = \Phi^A(F)$. Moreover, the enlarged system admits the strictly convex entropy:

$$\eta(v, F, Z, w) = \frac{1}{2}|v|^2 + G(F, Z, w)\,, \tag{3.8}$$

with corresponding flux

$$q_\alpha = v_i\,\frac{\partial G}{\partial \Xi^A}(\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\,. \tag{3.9}$$

We now discuss the various notions of solutions. A strong (or classical) solution is a $W^{2,\infty}$ function which satisfies (3.1); its derivatives automatically verify (3.2)-(3.3) and the strong form of the conservation of energy (3.5). A weak entropy solution is a weak solution of (3.1) which verifies

(3.5) as an inequality. In order to make sense of the weak forms the integrability of all quantities which appear has to be guaranteed.

The notion of measure valued solution that we use is motivated by the form of the extended system (3.6)-(3.7) and the existence theory of measure-valued solutions developed in [9]. A measure valued solution will consist of a map $y : Q \times \mathbb{R}^+ \to \mathbb{R}^3$, with distributional derivatives $F = \nabla y \in L^\infty(L^p)$, $v = \partial_t y \in L^\infty(L^2)$, and a Young measure $\boldsymbol{\nu} = (\boldsymbol{\nu}_{(x,t)})_{(x,t)\in\overline{Q}_T}$ generated by a sequence satisfying

$$\sup_{\epsilon,t} \int \eta(v^\varepsilon, F^\varepsilon, Z^\varepsilon, w^\varepsilon)\, dx \, < \infty$$

which represents weak limits in the following way:

$$\begin{aligned} &\text{wk-}\lim_{\epsilon\to 0} f(v^\varepsilon, F^\varepsilon, Z^\varepsilon, w^\varepsilon) = \int f(\lambda_v, \lambda_\Xi) d\boldsymbol{\nu}_{(x,t)}(\lambda_v, \lambda_\Xi) \\ &\qquad \forall \text{ continuous } f = f(\lambda_v, \lambda_\Xi) \text{ with } \lim_{|\lambda_v|+|\lambda_\Xi|\to\infty} \frac{f(\lambda_v, \lambda_\Xi)}{\frac{1}{2}|\lambda_v|^2 + G(\lambda_\Xi)} = 0 \end{aligned} \tag{3.10}$$

where $\lambda_v \in \mathbb{R}^3$, $\lambda_\Xi = (\lambda_F, \lambda_Z, \lambda_w) \in \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} = \mathbb{R}^{19}$. The Young measure is connected with the map y through the requirements that (almost everywhere)

$$F = \langle \boldsymbol{\nu}, \lambda_F \rangle\,, \quad v = \langle \boldsymbol{\nu}, \lambda_v \rangle\,, \quad \Xi = \langle \boldsymbol{\nu}, \lambda_\Xi \rangle\,. \tag{3.11}$$

The action of the Young measure is well defined on all functions that grow slower than the energy norm. This is the natural framework under the existence of energy norm bounds. With this in mind we define:

Definition 3.1 *A* measure-valued solution *to* (3.1) *consists of a map y, with distributional time and space derivatives $(v, F) \in L^\infty(L^2) \oplus L^\infty(L^p)$ and a Young measure $\boldsymbol{\nu} = (\boldsymbol{\nu}_{x,t})_{x,t\in\overline{Q}_T}$ as just described, such that for $i = 1, \dots 3$*

$$\partial_t v_i - \partial_\alpha \big\langle \boldsymbol{\nu}, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) \big\rangle = 0 \tag{3.12}$$

and for $A = 1, \dots 19$

$$\partial_t \Phi^A(F) - \partial_\alpha \big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \big) = 0 \tag{3.13}$$

in distributions with

$$\Xi = \Phi(\langle \boldsymbol{\nu}, \lambda_F \rangle) = \Phi(F)\,. \tag{3.14}$$

The solution is said to be a dissipative measure-valued solution with concentration *if it is a measure-valued solution which verifies in addition:*

$$\iint \frac{d\theta}{dt} \Big(\langle \boldsymbol{\nu}, \eta \rangle + \boldsymbol{\gamma} \Big)\, dxdt + \int \theta(0) \eta_0(x) dx \geq 0\,, \tag{3.15}$$

for all non-negative functions $\theta = \theta(t) \in C^1_c[0, T)$ with $\theta \geq 0$. Here η_0 means the entropy η evaluated on the initial data and $\boldsymbol{\gamma}$ is the non-negative concentration measure defined in section A.2.

The measure-valued solution satisfies the momentum equation (3.6) in the averaged (with respect to the Young measure) sense, but the constraint equation (3.7) in the classical weak sense. This is due to the weak continuity of the null-Lagrangians (see [4], [9, lemma 3]) and the weak continuity of the transport identities (3.3) which follows from the equation $\partial_t F = \nabla v$ for functions $v \in L^\infty(L^2)$, $F \in L^\infty(L^p)$ with $p > 4$, [9, lemmas 4 and 5].

The existence of a measure-valued solution satisfying (3.12)-(3.14) is proved in [9, Section 3] under the following hypotheses on the function G:

(H1) $G \in C^3(\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R}; [0,\infty))$ is a strictly convex function satisfying for some $\gamma > 0$ the bound $D^2 G \geq \gamma > 0$.

(H2) $G(F, Z, w) \geq c_1(|F|^p + |Z|^q + |w|^r + 1) - c_2$ where $p \in (4, \infty)$, $\;q, r \in [2, \infty)$.

(H3) $G(F, Z, w) \leq c(|F|^p + |Z|^q + |w|^r + 1)$

(H4) $|\partial_F G|^{\frac{p}{p-1}} + |\partial_Z G|^{\frac{p}{p-2}} + |\partial_w G|^{\frac{p}{p-3}} \leq C(|F|^p + |Z|^q + |w|^r + 1)$

The function

$$\bar{G} = \alpha|F|^p + \beta|Z|^q + \gamma|w|^r + |F|^2 + |Z|^2 + w^2 \tag{3.16}$$

verifies (H1)-(H3). It will also verify (H4) under the restrictions $p \geq 2q \geq 4$, $p \geq 3r \geq 6$.

Theorem 3.2 *Let G satisfy $(H1) - (H4)$. Given initial data $(v^0, F^0) \in L^2 \oplus L^p$, $p \geq 4$, there exists a dissipative measure-valued solution to* (3.12)-(3.15) *in the sense of definition 3.1.*

Proof The existence of a measure-valued solution is the main theorem in [9]. The fact that this solution satisfies (3.15) is proved by using the Young measure representation with concentration from section A.2 to take the limit of equation (3.16) in [9], using the piecewise constant interpolates v^h, ξ^h defined in (4.3) in [9], which generate the Young measure $\boldsymbol{\nu}$ in the solution. Using these definitions equation (3.16) in [9] implies that

$$\int_h^\infty \frac{\theta(t+h) - \theta(t)}{h} \int \eta(v^h, \xi^h)\, dx\, dt + \frac{1}{h}\int_0^h \theta(t+h)\, dt \int \eta(v^h(x,0), \xi^h(x,0))\, dx \geq 0$$

for all non-negative functions $\theta(t) \in C^1_c([0,T))$. We know that $\frac{\theta(t+h)-\theta(t)}{h} \to \dot{\theta}(t)$ uniformly as $h \to 0$, But since $\int \eta(v^h, \xi^h)\, dx$ is uniformly bounded this implies that in this limit we can replace $\frac{\theta(t+h)-\theta(t)}{h}$ by $\dot{\theta}(t)$, and then applying (A.4) we obtain (3.15). $\square$

The next objective is to prove the measure-valued-strong uniqueness theorem. In fact the uniqueness theorem applies to a slightly more general class of nonlinearities: we retain the hypotheses (H1)-(H3) on G, but replace (H4) by the (slightly) weaker hypothesis

(H4)$'$ $|\partial_F G| + |\partial_Z G|^{\frac{p}{p-1}} + |\partial_w G|^{\frac{p}{p-2}} \leq o(1)(|F|^p + |Z|^q + |w|^r + 1)$ where $o(1) \to 0$ as $|\Xi| \to \infty$.

A hypothesis like (H4)$'$ is necessary in order to represent the weak limit of the Piola-Kirchhoff stress $g_{i\alpha} = \frac{\partial G}{\partial \Xi^A}(\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)$. To this end notice that

$$\begin{aligned} \frac{|g_{i\alpha}|}{G(\Xi)} &= \frac{1}{G(\Xi)}\Big|\frac{\partial G}{\partial \Xi^A}(\Xi)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\Big| \\ &\leq \frac{|\partial_F G| + |\partial_Z G||F| + |\partial_w G||F|^2}{|F|^p + |Z|^q + |w|^r + 1} = o(1) \qquad \text{as } |\Xi| \to \infty\,. \end{aligned} \tag{3.17}$$

The last inequality follows from (H4)$'$ and Young's inequality $ab \le \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$, $a, b \ge 0$, $\frac{1}{p} + \frac{1}{p'} = 1$. By (3.17) and (3.10) the average Piola-Kirchhoff stress $< \boldsymbol{\nu}, g_{i\alpha} >$ is then a well defined locally integrable function which is the weak L^1 limit of $g_{i\alpha}$ evaluated along an approximating sequence. As an example notice that the function $\bar{G}$ in (3.16) will satisfy (H4)$'$ provided $p > q \ge 2$ and $p > 2r \ge 4$. We prove:

Theorem 3.3 *Let G satisfy $(H1)-(H3)$, $(H4)'$ and let $(y, \boldsymbol{\nu}, \boldsymbol{\gamma})$ be a dissipative measure-valued solution in the sense of definition 3.1. If the initial data equal those of a Lipschitz bounded solution $(\hat{v}, \hat{F}) \in W^{1,\infty}(\overline{Q}_T)$:*

$$(v(x,0), \Xi(x,0)) = (\hat{v}(x,0), \Phi(\hat{F}(x,0)))$$

then $\boldsymbol{\gamma}$ is zero, $(v, \Xi) = (\hat{v}, \Phi(\hat{F}))$ and $\boldsymbol{\nu} = \delta_{\hat{v}, \Phi(\hat{F})}$.

Proof The proof is based on a generalization of the relative entropy computation to the polyconvex case. Let $(y, \boldsymbol{\nu})$ the measure-valued solution with v, Ξ as in (3.11), and let $\hat{v}$, $\hat{\Xi} := \Phi(\hat{F})$ be the Lipschitz solution satisfying (3.2)-(3.3). As explained in section 2.2 we may take the test functions in (3.12) and (3.13) to be Lipschitz functions which vanish for large time. To start with subtract the weak form of the equations of motion for the measure-valued and the Lipschitz solutions: for $i = 1, \dots 3$

$$\begin{aligned} &\int \psi(x,0)(v_i - \hat{v}_i)(x,0)\, dx + \iint (v_i - \hat{v}_i)\partial_t \psi \, dxdt \\ &\quad = \iint \Big(\Big\langle \boldsymbol{\nu}, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) \Big\rangle - \frac{\partial G}{\partial \Xi^A}(\hat{\Xi}) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F}) \Big) \partial_\alpha \psi \, dxdt \end{aligned} \tag{3.18}$$

and for $A = 1, \dots 19$

$$\begin{aligned} &\int \psi(x,0)\big(\Xi^A(x,0) - \hat{\Xi}^A(x,0)\big)\, dx + \iint (\Xi^A - \hat{\Xi}^A)\partial_t \psi dxdt \\ &\quad = \iint \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F}) \hat{v}_i \Big) \partial_\alpha \psi \, dxdt \end{aligned} \tag{3.19}$$

where ψ is a Lipschitz test function that vanishes for sufficiently large times. Now choose ψ in (3.18) to be $\theta(t)\hat{v}_i$, and in (3.19) to be $\theta(t)\frac{\partial G}{\partial \Xi^A}(\Phi(\hat{F}))$, where $\theta \in C^1_c([0,T))$. Adding the resulting equations leads to the identity:

$$\begin{aligned} &\int \theta(0) \Big[\hat{v}_i(x,0)(v_i - \hat{v}_i)(x,0) + \big(\frac{\partial G}{\partial \Xi^A}(\hat{\Xi}^A)\big(\Xi^A - \hat{\Xi}^A\big)\big)(x,0) \Big]\, dx \\ &\quad + \iint \Big[(v_i - \hat{v}_i)\hat{v}_i + (\Xi^A - \hat{\Xi}^A) \frac{\partial G}{\partial \Xi^A}(\hat{\Xi}) \Big] \partial_t \theta \, dxdt \\ &= - \iint \Big[(v_i - \hat{v}_i)\partial_t \hat{v}_i + \big(\Xi^A - \hat{\Xi}^A\big) \partial_t \big(\frac{\partial G}{\partial \hat{\Xi}^A}(\hat{\Xi}) \big) - \partial_\alpha \hat{v}_i \, \Big\langle \boldsymbol{\nu}, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) \Big\rangle \\ &\qquad + \partial_\alpha \hat{v}_i \frac{\partial G}{\partial \Xi^A}(\hat{\Xi}) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F}) - \partial_\alpha \big(\frac{\partial G}{\partial \Xi^A}(\hat{\Xi}) \big) \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F}) \hat{v}_i \Big) \Big] \theta \, dxdt \end{aligned}$$

We now calculate, using the fact that $(\hat v,\hat F)$ is a classical solution of (3.6)-(3.7), and obtain the following identities for the quantity in square brackets:

$$
\begin{aligned}
I &:= (\partial_t \hat v_i)(v_i-\hat v_i) + \partial_t\Big(\frac{\partial G}{\partial \widehat{\Xi}^A}(\hat\Xi)\Big)(\Xi^A-\hat\Xi^A)\\
&\quad - \partial_\alpha \hat v_i\,\Big(\Big\langle \boldsymbol{\nu}, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F)\Big\rangle - \frac{\partial G}{\partial \Xi^A}(\hat\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat F)\Big)\\
&\quad - \partial_\alpha\big(\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\big)\,\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat F)\hat v_i\Big)\\
&= -(\partial_\alpha \hat v_i)\Big[\Big\langle \boldsymbol{\nu}\,, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F)\Big\rangle - \frac{\partial G}{\partial \Xi^A}(\hat\Xi)\frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}\\
&\qquad\qquad - \frac{\partial^2 G}{\partial \Xi^A \partial \Xi^B}(\hat\Xi))\frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}(\Xi^B-\hat\Xi^B)\Big]\\
&\quad - \partial_\alpha\big(\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\big)\,\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\,v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat F)\,\hat v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat F)\,(v_i-\hat v_i)\Big)\\
&= -(\partial_\alpha \hat v_i)\frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}\Big\langle \boldsymbol{\nu}\,, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) - \frac{\partial G}{\partial \Xi^A}(\hat\Xi) - \frac{\partial^2 G}{\partial \Xi^A\partial \Xi^B}(\hat\Xi)(\lambda_{\Xi^B}-\hat\Xi^B)\Big\rangle\\
&\quad - (\partial_\alpha \hat v_i)\Big\langle \boldsymbol{\nu}\,, \Big(\frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) - \frac{\partial G}{\partial \Xi^A}(\hat\Xi)\Big)\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat F)\Big)\Big\rangle\\
&\quad - \partial_\alpha\big(\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\big)\Big(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}\Big)(v_i-\hat v_i)\\
&\quad - (\partial_\alpha \hat v_i)\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\Big\langle \boldsymbol{\nu}\,, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat F)\Big\rangle\\
&\quad - \partial_\alpha\big(\frac{\partial G(\hat\Xi))}{\partial \Xi^A}\big)\hat v_i\Big(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}\Big)
\end{aligned}
\tag{3.20}
$$

Using the fact that $<\boldsymbol{\nu}, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F)>= \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)$ and the null Lagrangian property (3.4), we see that the last two terms can be written as a divergence, and their contribution integrates to zero. For a test function $\theta \in C^1_c([0,T))$ we obtain:

$$
\begin{aligned}
&\int \theta(0)\Big[\hat v_i(x,0)(v_i-\hat v_i)(x,0) + \Big(\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\big(\Xi^A-\hat\Xi^A\big)\Big)(x,0)\Big]\,dx\\
&\quad + \iint \Big[(v_i-\hat v_i)\hat v_i + (\Xi^A-\hat\Xi^A)\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\Big]\,\partial_t\theta\,dxdt = \iint \mathcal{Q}\theta dxdt\,,
\end{aligned}
\tag{3.21}
$$

where $(-\mathcal{Q})$ stands for the first three terms in (3.20),

$$
\begin{aligned}
\mathcal{Q} &= (\partial_\alpha \hat v_i)\frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}\Big\langle \boldsymbol{\nu}\,, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) - \frac{\partial G}{\partial \Xi^A}(\hat\Xi) - \frac{\partial^2 G}{\partial \Xi^A\partial \Xi^B}(\hat\Xi)(\lambda_{\Xi^B}-\hat\Xi^B)\Big\rangle\\
&\quad (\partial_\alpha \hat v_i)\Big\langle \boldsymbol{\nu}\,, \Big(\frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) - \frac{\partial G}{\partial \Xi^A}(\hat\Xi)\Big)\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat F)\Big)\Big\rangle\\
&\qquad \partial_\alpha\big(\frac{\partial G}{\partial \Xi^A}(\hat\Xi)\big)\Big(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat F)}{\partial F_{i\alpha}}\Big)(v_i-\hat v_i)\\
&=: Q_1 + Q_2 + Q_3
\end{aligned}
\tag{3.22}
$$

Defining the relative entropy as

$$\eta_{rel}(v,\Xi;\hat{v},\hat{\Xi}) := \frac{1}{2}|v-\hat{v}|^2 + G(\Xi) - G(\hat{\Xi}) - \frac{\partial G}{\partial \Xi^A}(\hat{\Xi})\,(\Xi^A - \hat{\Xi}^A) \tag{3.23}$$

we prove that $\mathcal{Q}$ can be bounded by the averaged relative entropy:

Lemma 3.4 *Under Hypothesis $(H1)-(H3)$, $(H4)'$, there exists $C = C(|(\hat{v},\hat{\Xi})|_{W^{1,\infty}})$ such that*

$$|\mathcal{Q}| \le C\langle \boldsymbol{\nu}, \eta_{rel}\rangle\,, \quad \langle \boldsymbol{\nu}, \eta_{rel}\rangle = \int \eta_{rel}(\lambda_v, \lambda_\Xi; \hat{v}, \hat{\Xi})\,\boldsymbol{\nu}(d\lambda_v, d\lambda_\Xi)\,.$$

Proof of the lemma. We start by estimating the term Q_2 in (3.22). Let $K \subset \mathbb{R}^{19}$ be a compact set containing the values of $\hat{\Xi}(x,t)$ for $(x,t)\in Q_T$. We will show that there is a constant C such that for all $\lambda_\Xi \in \mathbb{R}^{19}$ and $\hat{\Xi}\in K$ there holds

$$|\mathcal{Q}_2| = \left|\Big(\frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) - \frac{\partial G}{\partial \Xi^A}(\hat{\Xi})\Big)\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F})\Big)\right| \le C G_{rel}(\lambda_\Xi;\hat{\Xi}), \tag{3.24}$$

where

$$G_{rel}(\lambda_\Xi;\hat{\Xi}) = G(\lambda_\Xi) - G(\hat{\Xi}) - D_\Xi G(\hat{\Xi})\cdot(\lambda_\Xi - \hat{\Xi}) \tag{3.25}$$

Note that the assumptions $(H1)-(H2)$ imply the lower bound

$$G_{rel}(\lambda_\Xi;\hat{\Xi}) \ge \max\{\gamma(|\lambda_\Xi - \hat{\Xi}|^2, \alpha(|\lambda_F|^p + |\lambda_Z|^q + |\lambda_w|^r + 1) - A\} \tag{3.26}$$

for some constants α, γ and A which depend upon $|\Xi|_{L^\infty}$ and the constants c_1, c_2 appearing in $(H1)-(H2)$.

Define now $\mathcal{L}_R = \{|\lambda_F|^p + |\lambda_Z|^q + |\lambda_w|^r + 1 \ge R\}$ with R chosen sufficiently large so that $K \subset (\mathcal{L}_R)^c$ and also

$$\alpha(|\lambda_F|^p + |\lambda_Z|^q + |\lambda_w|^r + 1) - A \ge \frac{\alpha}{2}(|\lambda_F|^p + |\lambda_Z|^q + |\lambda_w|^r + 1) \quad \text{ on } \mathcal{L}_R.$$

For $\lambda_\Xi \in \mathcal{L}_R$ and $\hat{\Xi}\in K$ we have upon using Young's inequality, hypothesis $(H4)'$, selecting R sufficiently large, and using (3.26) that

$$\begin{aligned}
|\mathcal{Q}_2| &\le C\Big[(1+|\partial_F G(\lambda_\Xi)|) + (1+|\lambda_F|)(1+|\partial_Z G(\lambda_\Xi)|) + (1+|\lambda_F|^2)(1+|\partial_w G(\lambda_\Xi)|)\Big] \\
&\le \frac{\alpha}{4}|\lambda_F|^p + C_\alpha\Big(|\partial_F G| + |\partial_Z G|^{\frac{p}{p-1}} + |\partial_w G|^{\frac{p}{p-2}}\Big) \\
&\le \frac{\alpha}{2}\big(|\lambda_F|^p + |\lambda_Z|^q + |\lambda_w|^r + 1\big) \\
&\le C G_{rel}(\lambda_\Xi;\hat{\Xi}) \qquad \lambda_\Xi \in \mathcal{L}_R\,,\ \hat{\Xi}\in K\,.
\end{aligned}$$

With R now fixed, observe that for $\lambda_\Xi \in (\mathcal{L}_R)^c$

$$\begin{aligned}
|\mathcal{Q}_2| &\le C_R|\lambda_\Xi - \hat{\Xi}|^2 \\
&\le \frac{C_R}{\gamma} G_{rel}(\lambda_\Xi;\hat{\Xi}) \qquad \lambda_\Xi \in (\mathcal{L}_R)^c\,,\ \hat{\Xi}\in K\,.
\end{aligned}$$

Therefore, (3.24) follows.

The term Q_1 is estimated using the bound

$$|\mathcal{Q}_1| = \Big|\frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) - \frac{\partial G}{\partial \Xi^A}(\hat{\Xi}) - \frac{\partial^2 G}{\partial \Xi^A \partial \Xi^B}(\hat{\Xi})(\lambda_{\Xi^B} - \hat{\Xi}^B)\Big| \leq C G_{rel}(\lambda_\Xi; \hat{\Xi}) \quad \lambda_\Xi \in \mathbb{R}^{19}, \hat{\Xi} \in K\,, \tag{3.27}$$

which follows from an argument similar to the derivation of (3.24) above (using the fact from H4′ that the derivatives of G grow more slowly than G itself at infinity).

Finally the term Q_3 is estimated using

$$|v - \hat{v}|^2 = \Big|\int (\lambda_v - \hat{v}) d\boldsymbol{\nu}\Big|^2 \leq \int |\lambda_v - \hat{v}|^2 d\boldsymbol{\nu} \leq C \int \eta_{rel}(\lambda_v, \lambda_\Xi; \hat{v}, \hat{\Xi})\, d\boldsymbol{\nu}(\lambda_v, \lambda_\Xi)\,, \tag{3.28}$$

the weak continuity property $< \boldsymbol{\nu}, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) >= \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)$ and the estimation (in the spirit of (3.24))

$$\Big|\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F})\Big|^2 \leq C G_{rel}(\lambda_\Xi; \hat{\Xi}) \quad \lambda_\Xi \in \mathbb{R}^{19}, \hat{\Xi} \in K\,.$$

Combining these we obtain

$$\begin{aligned}
\Big|\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F})\Big|^2 &= \Big|\int \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F})\Big) d\boldsymbol{\nu}\Big|^2 \\
&\leq \int \Big|\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\hat{F})\Big|^2 d\boldsymbol{\nu} \\
&\leq C \int G_{rel}(\lambda_\Xi; \hat{\Xi})\, d\boldsymbol{\nu}(\lambda_\Xi)\,,
\end{aligned}$$

and hence, by (3.28) and Cauchy-Schwarz,

$$|\mathcal{Q}_3| \leq C \int \eta_{rel}(\lambda_v, \lambda_\Xi; \hat{v}, \hat{\Xi})\, d\boldsymbol{\nu}(\lambda_v, \lambda_\Xi)\,. \tag{3.29}$$

The proof of the lemma is completed by refering to (3.27), (3.24) and (3.29). □

To conclude, from the definition of the dissipative measure valued solution (3.15) and the equations (3.21), (3.22), and lemma 3.4 we derive the equation for the relative entropy

$$\begin{aligned}
&\iint \frac{d\theta}{dt} \Big(\langle \boldsymbol{\nu}, \eta_{rel}\rangle\, dxdt + \boldsymbol{\gamma}(dxdt)\Big) \\
&\quad + \theta(0) \int \Big[\eta_0 - \hat{\eta}_0 - \hat{v}_i(v_i - \hat{v}_i) - \frac{\partial G}{\partial \Xi^A}(\widehat{\Xi}^A)\big(\Xi^A - \widehat{\Xi}^A\big)\Big]_{t=0} dx \geq -C \int \langle \boldsymbol{\nu}, \eta_{rel}\rangle\, dxdt\,,
\end{aligned} \tag{3.30}$$

for all $\theta = \theta(t) \in C_c^1([0, T))$, $\theta \geq 0$. The proof can now be completed as in the proof of theorem 2.5, leading to the bound

$$\int \langle \boldsymbol{\nu}, \eta_{rel}\rangle\, dx\,\Big|_t \leq C_2 e^{C_1 C_2 t} \int \Big[\eta_0 - \hat{\eta}_0 - \hat{v}_i(v_i - \hat{v}_i) - \frac{\partial G}{\partial \Xi^A}(\widehat{\Xi}^A)\big(\Xi^A - \widehat{\Xi}^A\big)\Big]_{t=0} dx\,.$$

This implies the uniqueness assertion in the theorem statement for appropriate initial data. □

4 Conservation laws with L^p bounds

In this section we consider a measure-valued solution for the system of n conservation laws (2.4) in the presence of L^p bounds for $1 < p < \infty$. We first show how to generalize theorem 2.2 on recovery of classical solutions to this case. We also discuss the problem of the initial trace, i.e. the sense in which a measure-valued solution assumes the initial data. In this latter regard we show that the presence of a convex entropy yields *strong* convergence of the averages $\frac{1}{\tau}\int_0^\tau v(\cdot,t)dt$ to the initial data, thus extending a result of DiPerna [10] to the L^p framework.

We assume that (2.1) is equipped with an entropy-entropy flux pair $\eta - q$ as in section 2.1 with the entropy η satisfying the hypotheses:

$$\eta \text{ positive and strictly convex, } D^2\eta \geq \gamma > 0 \tag{4.1}$$

$$\alpha\big(|\lambda|^p + 1\big) - A \leq \eta(\lambda) \leq C\big(|\lambda|^p + 1\big) \quad \lambda \in \mathbb{R}^n \tag{4.2}$$

for some $\alpha, \gamma, A, C > 0$ and for $p \in [2, \infty)$, while the flux f in (2.4) verifies the growth restriction

$$\frac{|f(\lambda)|}{\eta(\lambda)} = o(1) \quad \text{as } |\lambda| \to \infty\,. \tag{4.3}$$

The entropy identity provides stability in an L^p-framework, $p < \infty$. In contrast to the L^∞ case treated in section 2.1 such a framework permits the development of concentrations in approximating sequences, which we describe using the measure $\boldsymbol{\gamma}$ defined in appendix A. Using the Young-measure associated to the family $\{v^\varepsilon\}$ and the concentration measure $\boldsymbol{\gamma} \geq 0$ we have

$$g(v^\varepsilon) \rightharpoonup \langle \boldsymbol{\nu}_{x,t}, g(\lambda)\rangle\,, \quad \forall\, g \text{ continuous s.t. } \lim_{|\lambda|\to\infty} \frac{g(\lambda)}{\eta(\lambda)} = 0, \tag{4.4}$$

$$\eta(v^\varepsilon)\, dxdt \rightharpoonup \langle \boldsymbol{\nu}_{x,t}, \eta\rangle\, dxdt + \boldsymbol{\gamma}(dxdt) \tag{4.5}$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ in (4.5) are as introduced in appendix A.

For the initial data $\{v_0^\varepsilon\}$ of the approximating problem (2.2) we assume weak convergence to v_0 in L^p with associated Young measure $\boldsymbol{\mu}_x$, and also allow the development of concentrations in η described by a concentration measure $\boldsymbol{\zeta}(dx) \geq 0$ such that

$$g(v_0^\varepsilon) \rightharpoonup \langle \boldsymbol{\mu}_x, g(\lambda)\rangle\,, \quad \forall\, g \text{ continuous s.t. } \lim_{|\lambda|\to\infty} \frac{g(\lambda)}{\eta(\lambda)} = 0, \tag{4.6}$$

$$\eta(v_0^\varepsilon)\, dx \rightharpoonup \langle \boldsymbol{\mu}_x, g(\lambda)\rangle\, dx + \boldsymbol{\zeta}(dx)\,. \tag{4.7}$$

Definition 4.1 A *dissipative measure-valued solution with concentration* to (2.1) consists of $v \in L^\infty(L^p)$, a Young measure $(\boldsymbol{\nu}_{x,t})_{x,t\in\overline{Q}_T}$ and a non-negative Radon measure $\boldsymbol{\gamma} \in \mathcal{M}^+(Q_T)$ such that

$$\iint \langle \boldsymbol{\nu}_{x,t}, \lambda\rangle \psi_t\, dxdt + \iint \langle \boldsymbol{\nu}_{x,t}, f(\lambda)\rangle \psi_x\, dxdt + \int v_0(x)\psi(x,0)\, dx = 0 \tag{4.8}$$

for any $\psi \in C_c^1(Q \times [0,T))$, and

$$\iint \dot\theta\big(\langle \boldsymbol{\nu}_{x,t}, \eta(\lambda)\rangle\, dxdt\, + \boldsymbol{\gamma}(dxdt)\big) + \int \theta(0)\big(\langle \boldsymbol{\mu}_x, \eta\rangle\, dx + \boldsymbol{\zeta}(dx)\big) \geq 0\,, \tag{4.9}$$

for all $\theta = \theta(t) \in C_c^1([0,T))$ with $\theta \geq 0$.

4.1 Recovery of classical solutions from measure-valued solutions

We first state the generalization of theorem 2.2 in the L^p framework:

Theorem 4.2 *Let $(v, \boldsymbol{\nu}, \boldsymbol{\gamma})$ be a dissipative measure-valued solution as in definition 4.1, and suppose that there exists a strong solution $\overline{v} \in W^{1,\infty}(\overline{Q}_T)$ verifying* (2.11) *and* (2.12)*. If for the initial data $\boldsymbol{\zeta} = 0$ and $\boldsymbol{\mu}_x = \delta_{\overline{v}_0(x)}$ then $\boldsymbol{\nu} = \delta_{\overline{v}}$ and $v = \overline{v}$ almost everywhere on Q_T.*

Proof The initial calculations are identical to the L^∞ case in the proof of theorem 2.2 up to (2.22). Since the support of $\boldsymbol{\nu}$ is no longer bounded it is necessary to replace (2.24). This is done as follows: define $\eta_{rel}(\lambda, \overline{v})$ by (2.14) and let $K \subset \mathbb{R}^n$ be a compact set containing the values of $\bar{v}(x,t)$ for $(x,t) \in Q_T$. Using (4.1), (4.2), (4.3) and an argument as in the proof of (3.24) (see lemma 3.4), there exists a constant $C_1 > 0$ such that

$$\left|f_{k\alpha}(\lambda) - f_{k\alpha}(\overline{v}) - \frac{\partial f_{k\alpha}}{\partial v_j}(\overline{v})(\lambda_j - \overline{v}_j)\right| \leq C_1 \eta_{rel}(\lambda; \bar{v}) \qquad \lambda \in \mathbb{R}^n\,,\ \bar{v} \in K \tag{4.10}$$

and hence integrating over λ we obtain that

$$|Z_{k\alpha}(\boldsymbol{\nu}, v, \overline{v})| \leq C_1 h(\boldsymbol{\nu}, v, \overline{v})\,, \tag{4.11}$$

where we use the definitions (2.14)-(2.16). This inequality serves as a suitable replacement of (2.24) to complete the transposition of the proof of theorem 2.2 to the L^p setting: under the assumption $\boldsymbol{\zeta} = 0$ there holds

$$\int h(\boldsymbol{\nu}, v, \overline{v})\, dx \leq c_1 \int \eta_{rel}(\lambda, \overline{v}_0) d\boldsymbol{\mu}(\lambda)\, dx\ e^{c_2 t}\,, \tag{4.12}$$

and in particular if $v(x,0) = \overline{v}_0(x)$ and $\boldsymbol{\mu}_x = \delta_{\overline{v}_0(x)}$ then $\boldsymbol{\nu}_{x,t} = \delta_{\overline{v}(x,t)}$ and $v(x,t) = \overline{v}(x,t)$ for $t > 0$, and $\boldsymbol{\gamma} = 0$. □

4.2 On the initial trace of measure-valued solutions

DiPerna [10, section 6(e)] gave an argument indicating that the measure-valued version of the entropy condition, used in the case of strict convexity of the entropy, leads to a strong initial trace for a measure-valued solution in the L^∞ setting. Below this result is extended to the L^p functional setting, $p < \infty$.

Theorem 4.3 *Let v, $\boldsymbol{\nu}_{x,t}$ and $\boldsymbol{\gamma}(dxdt)$ be a dissipative measure-valued solution with concentration to* (2.1)*. If the Young measure associated with the data satisfies $\boldsymbol{\zeta} \equiv 0$ and $\boldsymbol{\mu}_x = \delta_{v_0(x)}$, then as $\tau \to 0+$*

$$\frac{1}{\tau}\int_0^\tau v(\cdot, t) dt \to v_0\,, \qquad \textit{in } L^p(Q). \tag{4.13}$$

Proof We first show that as a consequence of the definition of a measure-valued solution

$$\frac{1}{\tau}\int_0^\tau v(\cdot, t) dt \rightharpoonup v_0\,, \quad \text{weakly in } L^p(Q)\,. \tag{4.14}$$

To achieve this apply (4.8) to the test function $\psi(x,t) = \varphi(x)\theta(t)$, where $\varphi \in C^1(Q)$ and

$$\theta(t) \equiv \begin{cases} 1 - \frac{t}{\delta} & \text{when} \quad 0 \leq t \leq \delta\,, \\ 0 \ \text{ when } \delta \leq t\,. \end{cases} \tag{4.15}$$

Then we obtain

$$-\frac{1}{\delta}\int_0^\delta\int_Q v(x,t)\varphi(x)dxdt+\int_0^\delta\int_Q\langle\boldsymbol{\nu}_{x,t},f(\lambda)\rangle\varphi(x)\theta(t)dxdt$$
$$+\int_Q v_0(x)\varphi(x)dx=0\ .$$

Passing to the limit $\delta\to 0$, we conclude

$$\lim_{\delta\to 0}\int_Q\left(\frac{1}{\delta}\int_0^\delta v(x,t)dt\right)\varphi(x)dx\to\int v_0(x)\varphi(x)dx\ . \tag{4.16}$$

Since

$$\int_Q\left|\frac{1}{\delta}\int_0^\delta v(x,t)dt\right|^p dx\le\frac{1}{\delta}\int_Q\int_0^\delta|v|^p dxdt\le\|v\|_{L^\infty(L^p)} \tag{4.17}$$

equation (4.16), together with an approximation argument, implies that the sequence $\left\{\frac{1}{\delta}\int_0^\delta v(\cdot,t)dt\right\}$ converges weakly to v_0 in $L^p(Q)$.

Consider now the functional $I:L^p(Q)\to\mathbb{R}$ defined by

$$I[v]=\int_Q\eta(v)dx\,.$$

Due to the convexity of η the functional I is weakly lower semicontinuous. Hence (4.14) implies

$$\int_Q\eta(v_0(x))dx\le\liminf_{\delta\to 0}\int_Q\eta\left(\frac{1}{\delta}\int_0^\delta v(x,t)dt\right)dx \tag{4.18}$$

Fix θ as in (4.15) and consider a sequence of C^1 functions $\theta_n\to\theta$ that are monotone decreasing, vanish for large t, and satisfy $\theta_n(0)=1$ and $\dot\theta_n(t)\to\dot\theta(t)$ for $t\neq 0,\delta$. We apply (4.9) to the test functions θ_n and use the hypotheses for the initial measure and the property $\boldsymbol{\gamma}\ge 0$ to obtain

$$\int_Q\eta(v_0(x))dx\ge-\iint\frac{d\theta_n}{dt}\langle\boldsymbol{\nu}_{x,t},\eta(\lambda)\rangle dxdt\,.$$

Passing to the limit $n\to\infty$ and then $\delta\to 0$ and using $v(x,t)=\int\lambda d\boldsymbol{\nu}_{x,t}(\lambda)$ and Jensen's inequality we conclude that

$$\begin{aligned}\int_Q\eta(v_0(x))dx&\ge\limsup_{\delta\to 0}\frac{1}{\delta}\int_0^\delta\int_Q\int\eta(\lambda)d\boldsymbol{\nu}_{x,t}(\lambda)dxdt\;=\;\limsup_{\delta\to 0}\int_Q\frac{1}{\delta}\int_0^\delta\int\eta(\lambda)d\boldsymbol{\nu}_{x,t}(\lambda)dtdx\\&\ge\limsup_{\delta\to 0}\int_Q\frac{1}{\delta}\int_0^\delta\eta\left(\int\lambda d\boldsymbol{\nu}_{x,t}(\lambda)\right)dtdx\;=\;\limsup_{\delta\to 0}\int_Q\frac{1}{\delta}\int_0^\delta\eta(v(x,t))dtdx\\&\ge\limsup_{\delta\to 0}\int_Q\eta\left(\frac{1}{\delta}\int_0^\delta v(x,t)dt\right)dx\,.\end{aligned} \tag{4.19}$$

In summary, for the family $\left\{v^\delta=\frac{1}{\delta}\int_0^\delta v(\cdot,t)dt\right\}$, we have $v^\delta\rightharpoonup v_0$ weakly in $L^p(Q)$ and

$$\lim_{\delta\to 0}\int_Q\eta\big(v^\delta(x)\big)dx=\int_Q\eta(v_0(x))dx\ . \tag{4.20}$$

We claim this implies

$$v^\delta = \frac{1}{\delta}\int_0^\delta v(\cdot,t)dt \to v_0\,, \quad \text{in } L^p(Q). \tag{4.21}$$

Indeed, by (4.17), the sequence $\{v^\delta\}$ is uniformly bounded in $L^p(Q)$. The results of section (A.6) imply that there exists an associated Young measure $\boldsymbol{\kappa}_x$ and a concentration measure $\boldsymbol{\epsilon}(dx) \geq 0$ such that

$$\eta(v^\delta) \rightharpoonup \int \eta(\lambda)d\boldsymbol{\kappa}_x(\lambda) + \boldsymbol{\epsilon}(dx) \tag{4.22}$$

Now (4.14) implies that $\int \lambda d\boldsymbol{\kappa}_x(\lambda) = v_0(x)$, so that by (4.20) and (4.22) we get

$$\int_Q \int \eta(\lambda)d\boldsymbol{\kappa}_x(\lambda)\,dx + \int_Q \boldsymbol{\epsilon}(dx) = \int_Q \eta(v_0(x))dx = \int_Q \eta\left(\int \lambda d\boldsymbol{\kappa}_x(\lambda)\right)dx\,.$$

Using Jensen's inequality

$$\eta\left(\int \lambda d\boldsymbol{\kappa}_x(\lambda)\right) \leq \int \eta(\lambda)d\boldsymbol{\kappa}_x(\lambda) \tag{4.23}$$

we conclude that the concentration measure $\boldsymbol{\epsilon} \equiv 0$, and that necessarily (4.23) holds as equality. The latter implies that $\boldsymbol{\kappa}_x = \delta_{v_0(x)}$ and completes the proof of (4.13). □

5 Application: one dimensional elastodynamics as the continuum limit of a lattice model

Here we investigate a spatially discrete lattice approximation to one dimensional elastodynamics. Apart from interest in the continuum limit, the purpose is to show that the use of the relative entropy method provides an efficient way of proving strong convergence theorems for approximation schemes: it is only necessary to verify that the approximation scheme generates a dissipative measure-valued solution. For simplicity as above we consider the periodic case so that the spatial domain is $Q = \mathbb{R}/2\pi\mathbb{Z}$ on which are located N atoms at the points $\{x_i(t)\}_{i=0}^{N-1}$, at time t, and continued periodically $x_{N+i}(t) = x_i(t) + 2\pi \ \forall i$ when convenient. We assume the existence of an equilibrium configuration in which the atoms form a one dimensional array (lattice) in which the i^{th} atom has reference location $X_i = \frac{2\pi i}{N} = \epsilon i$ so they are all separated by a distance $\epsilon \equiv \frac{2\pi}{N}$ from their nearest neighbours on either side. We write $I_\epsilon^i = \{X : X_i \leq X < X_{i+1}\}$ for the intervals into which the domain is sub-divided by the reference locations X_i.

We will assume the dynamics is determined by a natural Lagrangian system of the following form:

- each atom has identical mass $\epsilon\rho = \frac{2\pi}{N}\rho$ (so that the total mass is $2\pi\rho$), and the kinetic energy is $T = \frac{1}{2}\epsilon\rho\Sigma_i \dot{x}_i{}^2$;
- the potential energy is given by $V = \sum_{i=0}^{N-1} W(\frac{x_{i+1}-x_i}{\epsilon})$, where W is a strictly convex C^3 function such that $W''(u) \geq c_0 > 0$ and $W(u) \geq max(0, c_1|u|^p - c_2)$ for $c_i > 0$, $p \geq 2$ and $u \in \mathbb{R}$;
- $\lim_{|u|\to+\infty} \frac{W'(u)}{|u|^p} = 0$

- finally, the Lagrangian

$$L = T - V = \sum_{i=0}^{N-1} \frac{\epsilon\rho}{2}\dot{x}_i^2 - \epsilon W(\frac{x_{i+1}-x_i}{\epsilon}).$$

Thus we have the following *equation of motion*

$$\frac{d}{dt}(\epsilon\rho\dot{x}_i) \;=\; W'(\frac{x_{i+1}-x_i}{\epsilon}) - W'(\frac{x_i - x_{i-1}}{\epsilon}) \tag{5.1}$$

solutions of which have *energy* which is independent of time t:

$$\sum_{i=0}^{N-1}\Big[\frac{\epsilon\rho}{2}\dot{x}_i^2 + \epsilon W(\frac{x_{i+1}-x_i}{\epsilon})\Big] \;=\; E_0 \tag{5.2}$$

where E_0 is determined by the initial data. The system (5.1) has a *first order* in time formulation obtained by setting:

$$\begin{aligned} v_i &= \dot{x}_i \\ \rho\frac{dv_i}{dt} &= \frac{1}{\epsilon}W'(\frac{x_{i+1}-x_i}{\epsilon}) - W'(\frac{x_i-x_{i-1}}{\epsilon})\,. \end{aligned} \tag{5.3}$$

We are interested in studying the limit as $N \to \infty$, or equivalently $\epsilon \to 0$, of this system, and relating it to continuum elastodynamics. To this end we introduce by interpolation the following functions:

$$\begin{aligned} y^\epsilon(t,X) &= \sum_{i=0}^{N-1}\Big(x_i + \frac{1}{\epsilon}(X - i\epsilon))(x_{i+1}-x_i)\Big)\mathbb{1}_{I_\epsilon^i}(X) \\ \tilde{y^\epsilon}(t,X) &= \sum_{i=0}^{N-1} x_i\mathbb{1}_{I_\epsilon^i}(X) \end{aligned} \tag{5.4}$$

for $I_i^\epsilon = [i\epsilon,(i+1)\epsilon)$, as above. We will prove that these two functions have the same limit as $\epsilon \to 0$, and are thus lattice versions of the same macroscopic object. In fact they are lattice versions of the Eulerian description of an elastic continuum, which proceeds via a function $X \mapsto y(t,X)$ which gives the location in space of that infinitesimal part of the body whose reference location is the point X. It follows from the periodic continuation $x_{N+i}(t) = x_i(t) + 2\pi\ \forall i$ that $y^\epsilon(t, X+2\pi) = y^\epsilon(t,X) + 2\pi$ and similarly for $\tilde{y^\epsilon}$.

Lemma 5.1 *Assume we have for each $N \in \{1,2,\dots\}$ a set of initial data $\{(x_i(0),\dot{x}_i(0))\}_{i=0}^{N-1}$ such that the energy is uniformly bounded, so that (5.2) with $\epsilon = \frac{2\pi}{N}$ holds for some $E_0 < \infty$ independent of N. Then for each such ϵ the functions y^ϵ and $\frac{\partial y^\epsilon}{\partial t}$ are bounded continuous functions of t, X, and there exist a constant C depending on the energy and on the coercivity constants, $C = C(E_0,\rho,c_1,c_2)$ such that*

(i) $\sup_t\Big(\|\frac{\partial y^\epsilon}{\partial t}\|_{L^2(Q)} + \|\frac{\partial y^\epsilon}{\partial X}\|_{L^p(Q)} + \|\frac{\partial \tilde{y}^\epsilon}{\partial t}\|_{L^2(Q)}\Big) \le C.$

(ii) $\sup_t\|\tilde{y}^\epsilon - y^\epsilon\|_{L^p(Q)} \le C\epsilon\,.$

Proof Notice that $|\frac{X-i\epsilon}{\epsilon}\mathbb{1}_{I^i_\epsilon}(X)| \le 1$ everywhere. Therefore,

$$\|\frac{\partial y^\epsilon}{\partial t}\|^2_{L^2(Q)} = \|\sum_{i=0}^{N-1}\Big(\dot{x}_i + \frac{X-i\epsilon}{\epsilon}(\dot{x}_{i+1}-\dot{x}_i)\Big)\mathbb{1}_{I^i_\epsilon}\|^2_{L^2(Q)} \;\le\; 5\sum_{i=0}^{N-1}\epsilon\|\dot{x}_i\|^2_{L^2(Q)} \;\le\; C$$

and similarly for $\tilde{y}^\epsilon$. Next observe that $\frac{\partial y^\epsilon}{\partial X} = \sum_{i=0}^{N-1}\frac{x_{i+1}-x_i}{\epsilon}\mathbb{1}_{I^i_\epsilon}$ is bounded in L^p by (5.2) and our assumption on W, since $c_1\epsilon|\frac{x_{i+1}-x_i}{\epsilon}|^p \le \epsilon W(\frac{x_{i+1}-x_i}{\epsilon}) + c_2$. This completes the proof of (i) using the energy bound (5.2).

The second assertion also follows from (5.2) and (i) since

$$\tilde{y^\epsilon} - y^\epsilon \;=\; \sum_{i=0}^{N-1}\frac{(X_i - i\epsilon)}{\epsilon}(x_{i+1}-x_i)\mathbb{1}_{I^i_\epsilon} \;=\; \sum_{i=0}^{N-1}\frac{(X_i - i\epsilon)}{\epsilon}\mathbb{1}_{I^i_\epsilon}\frac{\partial y^\epsilon}{\partial X}\epsilon$$

which implies (ii) as $|\frac{X-i\epsilon}{\epsilon}\mathbb{1}_{I^i_\epsilon}(X)| \le 1$. $\square$

For clarity, define the variables for the first order formulation,

$$\begin{aligned} u^\epsilon(t,X) &= \frac{\partial y^\epsilon}{\partial X}(t,X) \\ v^\epsilon(t,X) &= \sum_{i=0}^{N-1}\dot{x}_i\mathbb{1}_{I^i_\epsilon} = \sum_{i=0}^{N-1}v_i(t)\mathbb{1}_{I^i_\epsilon} = \dot{\tilde{y}}^\epsilon\,. \end{aligned} \tag{5.5}$$

Then the equations of motion (5.1) in first order formulation (5.3) become respectively,

$$\begin{aligned} \epsilon\rho\frac{\partial v^\epsilon}{\partial t} &= W'(u^\epsilon(t,X)) - W'(u^\epsilon(t,X-\epsilon)) \\ \frac{\partial u^\epsilon}{\partial t} &= \frac{\partial v^\epsilon}{\partial X} - \frac{\partial}{\partial X}(\dot{\tilde{y}}^\epsilon - \dot{y}^\epsilon) \end{aligned} \tag{5.6}$$

which in weak form can be written as:

$$\begin{aligned} \int_0^\infty\int_{-\pi}^{\pi}\rho\frac{\partial\phi}{\partial t}v^\epsilon - \frac{1}{\epsilon}\Big(\phi(t,X+\epsilon)-\phi(t,X)\Big)W'(u^\epsilon(X))\;dXdt + \int_{-\pi}^{\pi}\rho\phi(X,0)v^\epsilon(X,0)\;dX \;&= 0 \\ \int_0^\infty\int_{-\pi}^{\pi}\Big(\frac{\partial\phi}{\partial t}u^\epsilon - \frac{\partial\phi}{\partial X}v^\epsilon - \frac{\partial^2\phi}{\partial X\partial t}(\tilde{y}^\epsilon - y^\epsilon)\Big)\;dXdt \;+\; \int_{-\pi}^{\pi}\phi(X,0)u^\epsilon(X,0)\;dX \;&= 0 \end{aligned} \tag{5.7}$$

for all $\phi \in C^2_c(Q_\infty)$.

As in lemma 5.1 bounds which are uniform in ϵ come from energy conservation, which in first order variables takes the form

$$\frac{\rho}{2}\|v^\epsilon(t,\cdot)\|^2_{L^2} + \int W(u^\epsilon(t,\cdot))dX \;=\; \frac{\rho}{2}\|v^\epsilon(0,\cdot)\|^2_{L^2} + \int W(u^\epsilon(0,\cdot))dX \;\le E_0 < \infty. \tag{5.8}$$

Thus $\sup_t\big(\int|u^\epsilon|^p dX + \int|v^\epsilon|^2 dX\big) \le C$. To take the limit of (5.7) we use the facts that $\frac{\phi(t,X+\epsilon)-\phi(t,X)}{\epsilon} \longrightarrow \frac{\partial\phi}{\partial X}$ uniformly (since ϕ is a test function) and $\tilde{y^\epsilon} - y^\epsilon \longrightarrow 0$ in L^p by lemma 5.1.

In the limit $\epsilon \to 0$ there is a Young measure $\boldsymbol{\nu}$ which represents weak limits of the sequence $(u^\epsilon, v^\epsilon) \rightharpoonup (u,v)$:

$$v = \int\lambda\,d\boldsymbol{\nu}(M,\lambda) \quad\text{and}\quad u = \int M\,d\boldsymbol{\nu}(M,\lambda)\,,$$

and of functions $g(u^\epsilon, v^\epsilon)$ which are L^1 precompact, so that in particular

$$\lim_{\epsilon\to 0}\int_0^\infty\int_{-\pi}^{\pi}\phi g(u^\epsilon,v^\epsilon)\,dXdt = \int_0^\infty\int_{-\pi}^{\pi}\phi\,\langle\boldsymbol{\nu},g\rangle\,dXdt \tag{5.9}$$

$$= \int_0^\infty\int_{-\pi}^{\pi}\int\phi\,g(M,\lambda)\,d\boldsymbol{\nu}(M,\lambda)\,dXdt \tag{5.10}$$

for all bounded ϕ which are 2π-periodic in X and vanish for large t. On the other hand for the energy density $\eta(u^\epsilon,v^\epsilon) = \frac{\rho}{2}(v^\epsilon)^2 + W(u^\epsilon)$ we only have L^1 boundedness, and the weak limit includes a concentration measure $\boldsymbol{\gamma}$:

$$\lim_{\epsilon\to 0}\int_0^\infty\int_{-\pi}^{\pi}\phi\eta(u^\epsilon,v^\epsilon)\,dXdt = \int_0^\infty\int_{-\pi}^{\pi}\phi\,\langle\boldsymbol{\nu},\eta\rangle\,dXdt + \int_0^\infty\int_{-\pi}^{\pi}\phi\,\boldsymbol{\gamma}(dXdt)$$

for $\phi\in C_c(Q_\infty)$. Consider initial data with the property that $(u^\epsilon(X,0),v^\epsilon(X,0)) \to (u(X,0),v(X,0))$ in $L^p\times L^2$, and $\int\eta(u^\epsilon(X,0),v^\epsilon(X,0))\,dX \to \int\eta(u(X,0),v(X,0))\,dX$. On account of the assumptions on W the limit $(u,v,\boldsymbol{\nu})$ is a dissipative measure-valued solution in the sense that:

$$\int_0^\infty\int_{-\pi}^{\pi}\Big(\rho v\,\partial_t\phi + \langle\boldsymbol{\nu},W'\rangle\,\partial_X\phi\Big)\,dXdt \;+\; \int_{-\pi}^{\pi}\rho\phi(X,0)v(X,0)\;dX \;=0 \tag{5.11}$$

$$\int_0^\infty\int_{-\pi}^{\pi}\Big(u\,\partial_t\phi - v\,\partial_X\phi\Big)\,dXdt \;+\; \int_{-\pi}^{\pi}\phi(X,0)u(X,0)\;dX \;=0\,, \tag{5.12}$$

for all $\phi\in C_c^1(Q_\infty)$, and

$$\int_0^\infty\int_{-\pi}^{\pi}\dot\theta(t)\big(\langle\boldsymbol{\nu},\eta\rangle\,dXdt + \boldsymbol{\gamma}(dXdt)\big) + \theta(0)\int_{-\pi}^{\pi}\eta(\overline{u}(X,0)\,,\overline{v}(X,0))\,dX \ge 0\,, \tag{5.13}$$

for non-negative $\theta\in C^1([0,\infty))$. (In fact the dissipative condition (5.13) holds as an equality.)

Now using the relative entropy method and the convexity assumption on W we can prove that in fact the convergence is strong and concentration free when a classical solution $(\overline{u},\overline{v})$ exists on $\overline{Q}_T$:

Theorem 5.2 *Assume that there is a pair of Lipschitz functions $(\overline{u},\overline{v})\in W^{1,\infty}(\overline{Q}_T)$ which satisfy the continuum limit equations:*

$$\begin{aligned}\int_0^\infty\int_{-\pi}^{\pi}\rho\frac{\partial\phi}{\partial t}\overline{v} + \frac{\partial\phi}{\partial X}W'(\overline{u}(X))\;dXdt + \int_{-\pi}^{\pi}\rho\phi(X,0)\overline{v}(X,0)\;dX \;&=0\\ \int_0^\infty\int_{-\pi}^{\pi}\Big(\frac{\partial\phi}{\partial t}\overline{u} - \frac{\partial\phi}{\partial X}\overline{v}\Big)\;dXdt + \int_{-\pi}^{\pi}\phi(X,0)\overline{u}(X,0)\;dX \;&=0\,,\end{aligned} \tag{5.14}$$

for all $\phi\in C_c^1(Q_T)$. Assume that there is a sequence of initial configurations of the lattice $\{(x_i(0),\dot x_i(0))\}_{i=0}^{N-1}$ with uniformly bounded energy, and such that the corresponding interpolated functions $(u^\epsilon(X,0),v^\epsilon(X,0))$, $\epsilon = \frac{2\pi}{N}$, converge strongly to $(\overline{u}(X,0),\overline{v}(X,0))$ in $L^p\times L^2$ and $\int\eta(u^\epsilon(X,0),v^\epsilon(X,0))\,dX \to \int\eta(\overline{u}(X,0),\overline{v}(X,0))\,dX$. Then (u^ϵ,v^ϵ), as defined in (5.4) and (5.5) from the solutions $\{(x_i(t),\dot x_i(t))\}_{i=0}^{N-1}$ of the microscopic model, converge strongly in $L^p\times L^2(Q_T)$ to the continuum limit $(\overline{u},\overline{v})$. Alternatively said, the Young measure $\boldsymbol{\nu}$ is a Dirac measure supported on $(\overline{u},\overline{v})$ and there is no concentration, i.e. the concentration measure $\boldsymbol{\gamma}$ is null.

Proof We define the relative entropy as $h(\boldsymbol{\nu}, u, v, \overline{u}, \overline{v}) = \langle \boldsymbol{\nu}, \eta_{rel} \rangle = \int \eta_{rel}(M, \lambda; \overline{u}, \overline{v})\, d\boldsymbol{\nu}(M, \lambda)$ with

$$\begin{aligned}\eta_{rel}(M, \lambda; \overline{u}, \overline{v}) &= \eta(M, \lambda) - \eta(\overline{u}, \overline{v}) - \overline{v}(\lambda - \overline{v}) - W'(\overline{u})(M - \overline{u}) \\ &= \frac{\rho}{2}(\lambda - \overline{v})^2 + W(M) - W(\overline{u}) - W'(\overline{u})(M - \overline{u})\,.\end{aligned}$$

Under the assumptions on W above there exists $C > 0$ such that

$$\frac{W'(M) - W'(\overline{u}) - W''(\overline{u})(M - \overline{u})}{W(M) - W(\overline{u}) - W'(\overline{u})(M - \overline{u})} \leq C$$

everywhere. (The number C depends upon the bounded region D in which $\overline{u}$ takes its values). Given this inequality and the assumption that the intial data converge to the initial data $(\overline{u}_0, \overline{v}_0)$ of the bounded Lipschitz solution $(\overline{u}, \overline{v})$ we then deduce, via a calculation analogous to that in (2.41)-(2.44), that

$$\int h(\boldsymbol{\nu}, u, v, \overline{u}, \overline{v})\, dX\big|_t \leq C' \int_0^t \int h(\boldsymbol{\nu}, u, v, \overline{u}, \overline{v})\, dX\big|_\tau d\tau$$

for $0 \leq t < T$, and hence that h and $\boldsymbol{\gamma}$ are zero almost everywhere for positive times for which the classical solution exists. This implies that $\boldsymbol{\nu}_{(X,t)} = \delta_{(u(X,t), v(X,t))}$ as previously, and hence that the convergence of (u^ϵ, v^ϵ) to (u, v) is strong and concentration free as claimed. □

A Appendix: An energy concentration measure for measure-valued solutions

In this appendix we summarize what we need about the Young measure description of oscillations and concentrations in weakly convergent sequences of functions $f^\epsilon(y) \in \mathbb{R}^m$ defined on the set $\overline{Q}_T$, writing y as the independent variable ($y = (t, x)$).

We consider two settings in which the Fundamental Therem of Young Measures, as found in Ball [2], applies: the L^∞ setting of section 2.1 and the L^p setting of sections 2.2 and 3. In the L^∞ setting the theorem attaches to a uniformly bounded sequence of functions on $\overline{Q}_T$ a subsequence, still written f^ϵ, and a parametrized Young measure (meaning a weak* measurable $\overline{Q}_T$-parametrized family of probability measures $\boldsymbol{\nu} = (\boldsymbol{\nu}_y)_{y \in \overline{Q}_T}$) such that for any continuous function $F : \mathbb{R}^m \to \mathbb{R}$

$$F(f^\epsilon) \rightharpoonup \langle \boldsymbol{\nu}\,, F \rangle \qquad \text{weak* in} \;\; L^\infty(\overline{Q}_T). \tag{A.1}$$

In the L^p setting, $1 < p < \infty$, a similar conclusion holds for any sequence of functions f^ϵ which are bounded in L^p: for continuous F such that $F(f^\epsilon)$ is L^1 weakly precompact there holds

$$F(f^\epsilon) \rightharpoonup \langle \boldsymbol{\nu}\,, F \rangle \qquad \text{weakly in} \;\; L^1(\overline{Q}_T). \tag{A.2}$$

This representation will generally not hold if L^1 *weakly precompact* is replaced by L^1 *bounded* because concentrations can develop. Various tools have been introduced to describe this such as biting convergence, the generalized concentration Young measure, microlocal defect measure, H-measure, varifold measure included, see references [11, 3, 14, 20, 1, 13] and [12, Section 1.D]. Here we introduce by hand a simple measure $\boldsymbol{\gamma}$ of concentration effects *in the energy* or other non-negative functions F of *critical growth*, that is functions such that $F(f^\epsilon)$ is bounded, but not necessarily weakly precompact, in L^1 (for example, $|f^\epsilon|^p$ of an L^p-bounded sequence). This measure $\boldsymbol{\gamma}$ is a sharpening of the weak* defect measure $\boldsymbol{\sigma}$ of Lions (described in ([16, Chapter 9]). In fact its existence follows as a particular case of a quite general result [1, Theorem 2.5]. However since we only need a rather special case - to describe the weak limit of a single non-negative function η - we give a simple direct proof from first principles.

We introduce this measure in two separate cases, first for illustrative purposes in the L^2 setting which applies in section 2.2, and then in the more general setting which is useful in the case of a polyconvex energy of section 3.

A.1 The L^2 case

We now consider the case $p = 2$ in more detail: let $f^\epsilon(y)$ converge weakly in L^2 to $f(y)$, and assume that $\int |f^\epsilon(y)|^2 dy \leq K < \infty$. Then by the previous discussion

$$\int_{Q_T} F(f^\epsilon)(y) w(y) dy \longrightarrow \int_{Q_T} \langle \boldsymbol{\nu}_y, F \rangle w(y) dy$$

for any $w \in L^\infty(\overline{Q}_T)$ and for any F satisfying $\lim_{|z|\to+\infty} \frac{|F(z)|}{1+|z|^2} = 0$, (since this implies that $F(f^\epsilon)$ is sequentially weakly precompact in L^1 by the criterion of de la Vallee Poussin.) For the function $F(z) = |z|^2$ itself, however, $y \mapsto |f^\epsilon(y)|^2 dy$ are weak* precompact in the space of non-negative Radon measures $\mathcal{M}^+(\overline{Q}_T)$, and the functions $|f^\epsilon|^2$ need not be weakly precompact in L^1 and as a result the Young measure representation in general fails.

In this context we define a defect measure by applying the Banach-Alaoglu theorem to the sequence $|f^\epsilon - f|^2$ to obtain a subsequential weak* limit $\boldsymbol{\sigma}$, which is a non-negative Radon measure,

$$\boldsymbol{\sigma}(\psi) = \iint \psi d\boldsymbol{\sigma} = \lim_{\epsilon\to 0} \iint \psi \, |f^\epsilon - f|^2 \, dxdt \,, \tag{A.3}$$

for all $\psi \in C(\overline{Q}_T)$. Alternatively, noting the identity $|f^\epsilon|^2 = |f^\epsilon - f|^2 + |f|^2 + 2\langle f, f^\epsilon - f\rangle$, it follows from the definition of weak L^2 convergence that an equivalent definition is

$$\boldsymbol{\sigma} = \text{wk*-}\lim_{\epsilon\to 0}(|f^\epsilon|^2 - |f|^2) \in \mathcal{M}^+(\overline{Q}_T).$$

Simple examples indicate that $\boldsymbol{\sigma}$ can be non-zero due to purely oscillatory effects, and it is "too large" to describe concentration effects in a useful way. Therefore we will use a modification of $\boldsymbol{\sigma}$, called $\boldsymbol{\gamma}$, which is smaller (i.e. $\boldsymbol{\gamma}(E) \leq \boldsymbol{\sigma}(E)$) and is designed to be useful to describe weak limits of non-negative functions of critical growth. To introduce the measure $\boldsymbol{\gamma}$ we first observe that if we apply the Young measure theorem to f^ϵ we obtain for almost every $y \in \overline{Q}_T$ a Radon probability measure $\boldsymbol{\nu}_y$, and the function $\int |\lambda|^2 \boldsymbol{\nu}_y(d\lambda)$ is well defined in the extended non-negatives $[0, \infty]$ by the monotone convergence theorem. Indeed let $q_R(\lambda) = |\lambda|^2 \mathbb{1}_{|\lambda \leq R} + R^2 \mathbb{1}_{|\lambda| \geq R}$ then $q_R(\lambda) \nearrow q(\lambda) = |\lambda|^2$ and so $\int |\lambda|^2 \boldsymbol{\nu}_y(d\lambda) = \lim_{R\to\infty} \int q_R(\lambda) \boldsymbol{\nu}_y(d\lambda)$ is well defined for a.e. y and is in $L^1(\overline{Q}_T)$ since by the Young measure representation theorem

$$\iint \psi(y) q_R(\lambda) \boldsymbol{\nu}_y(d\lambda) dy = \lim \int \psi(y) q_R(f^\epsilon(y)) dy \leq K \max_{y\in Q} |\psi(y)|$$

for all $\psi \in C(\overline{Q}_T)$; choosing $\psi(y) \equiv 1$ allows us to apply the monotone convergence theorem again to deduce that $\langle \boldsymbol{\nu}_y(\lambda), |\lambda|^2 \rangle = \int |\lambda|^2 \boldsymbol{\nu}_y(d\lambda) \in L^1(\overline{Q}_T)$ since it is a monotone non-decreasing limit of non-negative functions of uniformly bounded integral.

Now to define the concentration measure $\boldsymbol{\gamma}$, we just mimic the definition of σ, replacing $|f(y)|^2$ by $\langle \boldsymbol{\nu}_y(\lambda), |\lambda|^2\rangle$, i.e. we consider wk*-$\lim_{\epsilon\to 0}(|f^\epsilon(y)|^2 - \langle \boldsymbol{\nu}_y(\lambda), |\lambda|^2\rangle)$. To show that this limit exists in $\mathcal{M}^+(\overline{Q}_T)$ we use again the Young measure representation: for any $R > 0$ and any *non-negative* function $\psi \in C(\overline{Q}_T)$,

$$\iint \psi(y) q_R(\lambda) \boldsymbol{\nu}_y(d\lambda)\, dy = \lim_{\epsilon\to 0} \int \psi(y) q_R(f^\epsilon(y))\, dy \le \lim_{\epsilon\to 0} \int \psi(y) |f^\epsilon(y)|^2\, dy$$

and therefore

$$\iint \psi(y) |\lambda|^2 \boldsymbol{\nu}_y(d\lambda)\, dy = \sup_{R>0} \iint \psi(y) q_R(\lambda) \boldsymbol{\nu}_y(d\lambda)\, dy \le \lim_{\epsilon\to 0} \int \psi(y) |f^\epsilon(y)|^2\, dy$$

and hence

$$\boldsymbol{\gamma} = \text{wk*-}\lim_{\epsilon\to 0} \big(|f^\epsilon|^2 - \langle \boldsymbol{\nu}_y(\lambda), |\lambda|^2\rangle\big) \in \mathcal{M}^+(Q) \tag{A.4}$$

is a well defined *non-negative* Radon measure. Since Hölder's inequality implies that $|f(y)|^2 = |\langle \boldsymbol{\nu}_y, \lambda\rangle|^2 \le \langle \boldsymbol{\nu}_y, |\lambda|^2\rangle$, this definition implies that $\boldsymbol{\gamma} \le \boldsymbol{\sigma}$ as claimed earlier. The reason that the concentration Young measure $\boldsymbol{\gamma}$ is useful is that it allows a description of the weak limit of the energy, in terms of the Young measure $\boldsymbol{\nu}$ - the measure defined in (A.4) is used in section 2.2.

A.2 The general case

To describe concentration effects arising from more general energy functionals, such as the polyconvex ones in section 3, it is necessary to generalize the preceding definition. We now show that the same argument can be applied to any *non-negative* continuous function η which satisfies $\int \eta(f^\epsilon) \le K < \infty$, but for which the de la Vallee Poussin criterion does not apply and weak L^1 precompactness of $\eta(f^\epsilon)$ cannot be assumed. Instead we assume that $\eta \ge 0$ is a superlinear function and $\sup_{\epsilon>0} \int \eta(f^\epsilon) dx < K$ where f^ϵ is assumed to be a sequence of Lebesgue measurable functions which according to the theorem of Ball ([2]) has a subsequence, also called f^ϵ, with associated Young measure $\boldsymbol{\nu}_y$, which is a weak* measurable family of Radon *probability* measures on account of the superlinearity assumption on η. By the same theorem the Young measure represents L^1 weak limits of compositions of the f^ϵ as in (A.2) when these are L^1 precompact. Observe that

$$y \mapsto \int \eta(\lambda) \boldsymbol{\nu}_y(d\lambda)$$

is well defined a.e. in y with values in the extended non-negatives $[0, \infty]$ and is in L^1 by the monotone convergence theorem: $\eta_R(\lambda) = \eta(\lambda)\mathbb{1}_{\eta(\lambda)\le R} + R\mathbb{1}_{\eta(\lambda)\ge R}$ then $\eta_R(\lambda) \nearrow \eta(\lambda)$ and so $\int \eta(\lambda) \boldsymbol{\nu}_y(d\lambda) = \lim_{R\to\infty} \int \eta_R(\lambda) \boldsymbol{\nu}_y(d\lambda)$ is well defined for a.e. y and is in $L^1(\overline{Q}_T)$ since by the Young measure representation theorem

$$\iint \psi(y) \eta_R(\lambda) \boldsymbol{\nu}_y(d\lambda) dy = \lim_{\epsilon\to 0} \int \psi(y) \eta_R(f^\epsilon(y)) dy \le K \max_{y\in\overline{Q}_T} |\psi(y)|$$

for all $\psi \in C(\overline{Q}_T)$. Choosing $\psi(y) \equiv 1$ allows us to apply the monotone convergence theorem again to deduce that

$$\langle \boldsymbol{\nu}_y(\lambda), \eta(\lambda)\rangle = \int \eta(\lambda)) \boldsymbol{\nu}_y(d\lambda) \in L^1(\overline{Q}_T) \tag{A.5}$$

since it is a non-decreasing limit of non-negative functions of uniformly bounded integral: explicitly, by the Young measure representation for $\eta_R(f^\epsilon)$,

$$\int \langle \boldsymbol{\nu}_y(\lambda), \eta_R(\lambda)\rangle = \lim_{\epsilon\to 0}\int \eta_R(f^\epsilon) \ \leq \ \sup_\epsilon \int \eta_R(f^\epsilon) \ \leq \sup_\epsilon \int \eta(f^\epsilon) \leq K$$

by assumption on η and (f^ϵ) where the integrals are over Q_T and using that $0 < \eta_R \nearrow \eta$ we deduce (A.5) by monotone convergence taking the limit in R .

It is not, however, the case that $\eta(f^\epsilon)$ are L^1 precompact and so $\langle \boldsymbol{\nu}_y, \eta\rangle$ does not give its weak limit in general due to concentration. The concentration effect can be measured by considering the *concentration measure*

$$\boldsymbol{\gamma} = \text{wk*-}\lim_{\epsilon\to 0}(\eta(f^\epsilon) - \langle \boldsymbol{\nu}_y, \eta\rangle) \tag{A.6}$$

which is a well defined non-negative Radon measure for a subsequence of the $\eta(f^\epsilon)$ (since they have bounded integral): to see that $\boldsymbol{\gamma}$ is indeed non-negative we use again the Young measure representation to deduce that for any $R > 0$, and any *non-negative* function $\psi \in C(\overline{Q}_T)$,

$$\iint \psi(y)\eta_R(\lambda)\boldsymbol{\nu}_y(d\lambda)\,dy = \lim_{\epsilon\to 0}\int \psi(y)\eta_R(f^\epsilon(y))\,dy \leq \lim_{\epsilon\to 0}\int \psi(y)\eta(f^\epsilon(y))\,dy$$

and therefore

$$\iint \psi(y)\eta(\lambda)\boldsymbol{\nu}_y(d\lambda)\,dy = \sup_{R>0}\iint \psi(y)\eta_R(\lambda)\boldsymbol{\nu}_y(d\lambda)\,dy \leq \lim_{\epsilon\to 0}\int \psi(y)\eta(f^\epsilon(y))\,dy$$

and hence

$$\boldsymbol{\gamma} = \text{wk*-}\lim_{\epsilon\to 0}\big(\eta(f^\epsilon) - \langle \boldsymbol{\nu}_y, \eta\rangle\big) \in \mathcal{M}^+(\overline{Q}_T)$$

is a well defined *non-negative* Radon measure.

If in addition η is convex, then $\boldsymbol{\gamma} \leq \boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ is the natural generalization of the weak* defect measure, namely $\boldsymbol{\sigma} = \text{wk*-}\lim_{\epsilon\to 0}(\eta(f^\epsilon) - \eta(f)) \in \mathcal{M}^+(\overline{Q}_T)$. This is an immediate consequence of Jensen's inequality which implies that

$$\int \eta(\lambda)\ d\boldsymbol{\nu} \geq \eta(\int \lambda\ d\boldsymbol{\nu}) = \eta(\lim f^\epsilon) = \eta(f).$$

The reason that $\boldsymbol{\gamma}$ is useful is that it allows a description of the weak limit of the energy, in terms of the Young measure $\boldsymbol{\nu}$. In section 3 this applies to a sequence $f^\epsilon = (v^\epsilon, \Xi^\epsilon)$ which is bounded in a direct sum of different Lebesgue spaces, and which therefore has a weak limit point in the same space.

Remark A.1 *Although we refer to $\boldsymbol{\gamma}$ as concentration measure, it is not always supported on a small set: there exist sequences of functions in which the concentration smears out to fill the whole domain, see [3, Example 2].*

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